Cosmological Evolution of Energy Density and Power Density Perturbations

(Evolución cosmológica de perturbaciones de densidad de energía y de potencia)

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Abstract

Under the widely accepted consensus within Modern Cosmology, those structures visible today in the Universe, such as stars, galaxies, galaxy clusters and Large Scale Structure, form through the evolution, first lineal and then non-linear, of tiny primordial density fluctuation originated through quantum mechanisms during Cosmic Inflation. The evolution of those perturbations is governed by opposing processes of gravitational attraction, fluid pressure and cosmic expansion. Though these processes can be studied from a classical point of view, all of them are correctly described under the theoretical frame of General Relativity, If fluctuations are small enough, the equations describing their evolution can be linearized, giving way to the Linear Scalar Perturbation Theory currently used in Cosmology.

In this work we have studied the theoretical concepts of Linear Perturbation Theory, making extensive use of ideas from General Relativity such as *gauge invariance* or *perturbed Einstein Equations.* We show the derivation of the evolution equation for general linear scalar perturbations, carrying out all the intermediate steps needed. Under certain simplifications, we have obtained complete solutions in terms of the initial conditions, which motivates the introduction of the *power perturbations*, describing the instantaneous change in the density perturbations. Our solutions reproduce the behavior from the solutions obtained by the traditional treatment of the perturbations, with the novelty that they describe the evolution of perturbations of any scale, taking into account all the evolution modes (not only the dominant ones), and allow for the complete determination of the evolution given a set of initial conditions. Furthermore, a physical interpretation of the solution is conducted, as well as the evolution of their power spectrum coefficients, which could be related to observational data from the Cosmic Microwave Background and the distribution of Large Scale Structure.

Key words: Cosmological perturbations, General Relativity, gauge invariance, power spectrum, structure formation.

Resumen

Dentro del consenso ampliamente aceptado en la Cosmología Moderna, las estructuras hoy visibles en el Universo (estrellas, galaxias, cúmulos de galaxias, Estructura a Gran Escala) surgen por evolución, primero lineal y posteriormente no-lineal, de pequeñísimas fluctuaciones primordiales de densidad originadas mediante mecanismos cuánticos durante la Inflación Cósmica. La evolución de dichas perturbaciones viene regida por los efectos mutuamente contrapuestos de la atracción gravitatoria, la presión del fluido cósmico y la expansión del Universo que, si bien se pueden tratar de estudiar desde un punto de vista clásico, todos ellos están descritos adecuadamente dentro del marco teórico de la Teoría General de la Relatividad. Si las fluctuaciones son lo suficientemente pequeñas, las ecuaciones que describen su evolución pueden linealizarse, dando origen a la Teoría Lineal de Perturbaciones Escalares que se utiliza actualmente en Cosmología.

En este trabajo se han estudiado los conceptos teóricos de dicha Teoría Lineal de Perturbaciones Escalares, haciéndose amplio uso de conceptos de Relatividad General como *invariancia gauge* o las *Ecuaciones de Einstein Perturbadas.* Se muestra la derivación, llevándose a cabo todos los cálculos intermedios, de la ecuación que rige la evolución de perturbaciones escalares lineales de tipo general. Bajo ciertas simplificaciones, se han obtenido soluciones completas en función de las condiciones iniciales, para lo cual se hace necesario introducir *perturbaciones de potencia*, que describen la variaciones instantáneas de las perturbaciones de densidad. Estas soluciones reproducen los resultados obtenidos mediante el tratamiento tradicionalmente aceptado, con la novedad de que describen la evolución de perturbaciones de cualquier escala, teniendo en cuenta todos los modos de evolución, no únicamente los dominantes, y permiten determinar completamente la evolución de las perturbaciones en función de sus condiciones iniciales. Se lleva a cabo además una interpretación física crítica de las soluciones, así como la obtención de la evolución de los coeficientes del espectro de potencias de estas, a partir de los cuales se podría relacionar con los datos observacionales del Fondo Cósmico de Microondas y Estructura a Gran Escala.

Palabras Clave: Perturbaciones cosmológicas, Relatividad General, invarianza gauge, espectro de potencias, formación de estructura.

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CHAPTER 1

Introduction

When looking at the night sky with a telescope, apart from the Moon, the planets and the band of our own Galaxy, we observe approximately the same distribution of stars and galaxies in every direction, subject only to random statistical fluctuations. This is what we know as *isotropy*, meaning that our measurements do not depend on the observation direction.

If we now fix a certain direction and sort our observations with distance, we observe that further galaxies feature redder colors than they should based on their type, from where a Doppler-like behavior is inferred, with far astronomical objects drifting away from us. Apart from these, very distant objects, such as quasars, seem to concentrate at redshifts $1 \leq z \leq 3$ (see [**Pâris, 2018**]). This does not imply that the Universe has different and distinct regions, but it is rather a consequence of the finite nature of the speed of light, which means that the images from objects at great distances from us were actually emitted a long time ago, when the Universe was younger and different. Taking into account the effect of these two phenomenons, we could assume there is no difference between the properties of different points of the Universe. We call this property *homogeneity*. It can be easily shown that an isotropic Universe implies homogeneity, but not the other way around.

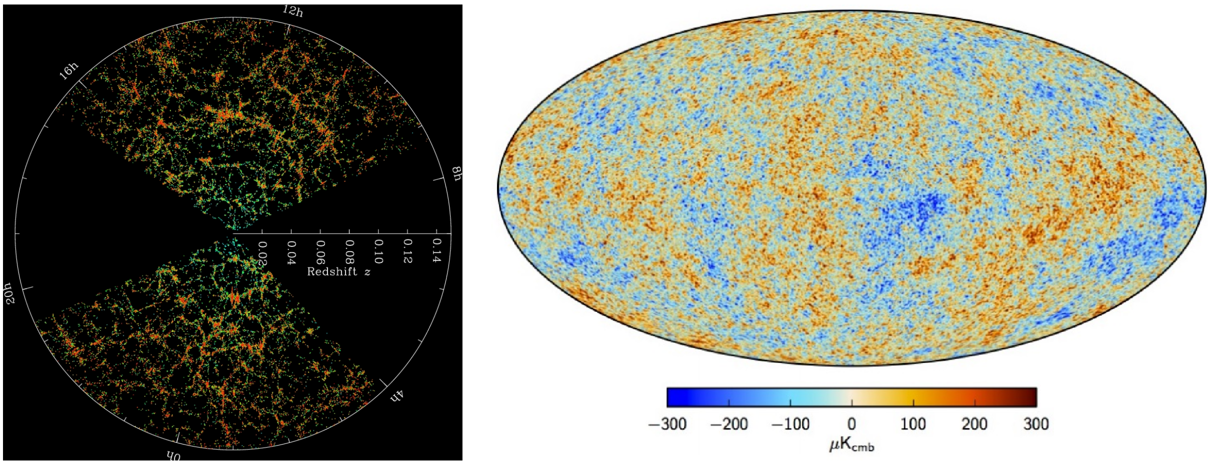

FIGURE 1. Results from the SDSS survey on local galaxies distribution, for a redshift between 0 and 0.15 (source: https://www.sdss.org/science/) (*left*). Difference between the average temperature of the CMB and the measured signal. Notice the scale of the differences ($\sim 10^{-5}$ K) compared to the average value of $T = 2.7255 \pm 0.0006$ K (source: https://www.cosmos.esa.int/web/planck/picture-gallery)(*right*).

While these two assumptions seem reasonable, they are backed by experimental evidence:

- **Hubble isotropy**: As it has been introduced in the previous paragraphs, astronomical objects located far away seem to have a receding velocity v from Earth, which seems to be directly proportional to their distance D, $v = H_0 D$, where $H_0 = 68.34$ kms^{-1}Mpc^{-1} is usually called *Hubble constant* (which actually is not a constant as it changes with time). This effect can be explained, as we shall describe in following sections, by the expansion of the Universe. If our Universe were isotropic, it would expand at the same rate in every direction, which would mean that the measured recession velocity would not depend on direction. Recent studies ([**Migkas, 2016**]) seem to support this.
- **Cosmic Microwave Background (CMB)**: Sensitive enough radio telescopes are able to measure a faint background electromagnetic signal, almost perfectly matching a black body thermal spectrum with $T = 2.7255 \pm 0.0006$ K. This signal does not correspond to any astronomical object such as stars, but rather exists as a relic radiation of the *Epoch of Recombination*, when temperatures started being low enough for photons and baryons to decouple, so light could travel freely. After subtracting the Doppler shift caused by the peculiar motion of the Solar system, the CMB is found to be *extremely* isotropic, with $\sigma_T = 18$ μK.
- **Galaxy distribution and Large-scale Structure**: Using sky surveys of galaxy clusters, the distribution of galaxies becomes more isotropic and uniform as scales grow larger, if we take into account

that the fact that the further the objects are measured, the older is the signal reaching us. It must be noted that as we measure further objects, two phenomenons overlap. On one hand they correspond to earlier objects in the history of the Universe, while on the other their signal is weaker and more difficult to measure, which translated in fewer detected sources.

As we shall review in the following chapter, under assumptions of homogeneity and isotropy, the overall evolution of the Universe can be easily obtained from a compact set of expressions, namely the Friedmann and the Continuity Equations, describing both the general geometrical (expansion) and dynamical (pressure and density) changes that it undergoes.

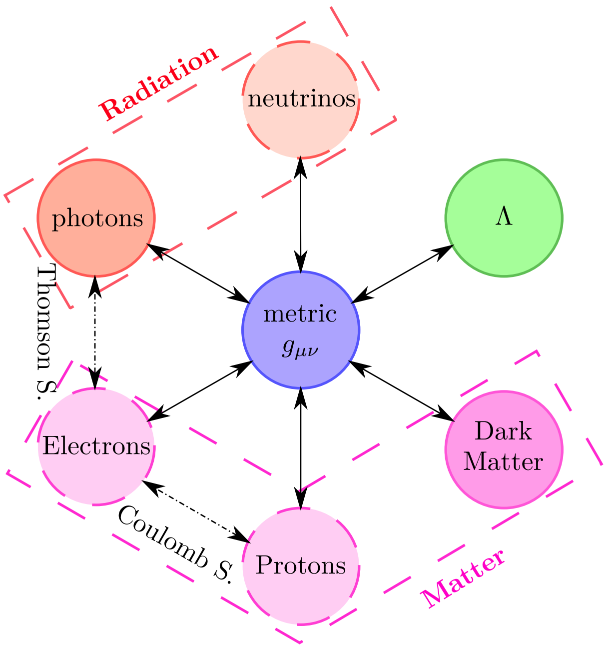

Figure 2. Schematic representation of the different relations between the different components of the Universe, and how they influence one another through the metric and other processes. Those elements not taken into consideration in the general discussion of this work appear in a lighter, dashed circle. Adapted from [**Baumann, 2016**]

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However, as evidenced by the anisotropies in the CMB or the existence of Large-scale structure, the Universe is not perfectly homogeneous. These "imperfections", while small enough so that the overall Universe can still be considered homogeneous and isotropic, are the reason galaxies, planets, and ultimately Humanity exists. The field of cosmological perturbations is a central aspect of Modern Cosmology, as serves as link between the different theoretical models trying to explain the origin of the Universe as well as the primordial inhomogeneities, and the high-precission Observational Astronomy and Cosmology, through which the distribution of the large-scale of the Universe at late times and the anisotropies of the CMB (produced at earlier times) are measured.

These inhomogeneities, if small enough, allow for a perturbative approach in order to obtain their evolution. Corresponding to under- or over-densities on the background mean density $\bar{\rho}$ of the Universe, it is possible to approach this problem using classical, Newtonian dynamics and gravity. However, in order to take into account fully relativistic effects such as the expansion of the Universe or space-time curvature, a complete covariant approach is needed, by the way of the Einstein Equations, $G_{\mu\nu} = 8\pi G c^{-4} T_{\mu\nu}$, which relate the presence of matter and energy in the Universe with its local metric properties, such as curvature. While these "gravitational" effects affect all the components of the Universe, other kinds of interactions also can be considered, as depicted in Figure 2.

1.1. Objectives and structure of this work

In order to clarify the purpose of this work, it is important to set our principle goals and objectives,which can be divided in two.

- After a brief exposition of the undergraduate-level theoretical background needed in **Chapter 2**, a detailed discussion of the *Linear Perturbation Theory* is given in **Chapter 3**, discussing the subtleties of working with cosmological perturbations in General Relativity. We will derive a *gauge-invariant* second order evolution equation for scalar density perturbations, known as *Bardeen's Equation* (originally published in [**Bardeen, 1980**]), going through all the intermediate steps, performing the necessary calculations and and giving physical interpretation to the various magnitudes an expression, something the original paper lacks. A basic knowledge of (pseudo-)Riemannian Geometry is expected in order to follow this chapter, so a brief review on the matter is given on **Appendix B**.
- Along **Chapter 4**, we will work with the obtained evolution equation and, performing some simplifications, we will derive a *complete* analytic expression for the density perturbations δ_M under different equations of state ω. Our approach is innovative in two ways. Not only we will obtain all the evolution

terms taking part in the evolution (the usual technique consists in performing several simplifications that allow to an easier solving of the evolution equations, but in which only the growing, dominant terms are obtained), but we will also express it in terms of the initial conditions of the perturbations. As the equation to solve is a second order ODE, two initial conditions are required. Other novel element of our work is that we will obtain also the evolution of the *power perturbations*, γ_M, expressing the instantaneous variation of the perturbations, and which directly take part as initial conditions of the solutions. Additionally, our solutions are valid for all scales, both sub- and superhorizon (these concepts will be discussed at the end of Chapter 3), and will allow us to obtain values for different phenomena which are not possible to calculate through more traditional approaches. We will describe the evolution of both the perturbations $\{\delta_M, \gamma_M\}$ and the coefficients of the associated power spectra $\{\Delta^2_{\delta_M}, \Delta^2_{\gamma_M}, \Xi\}$. The theory behind the statistical significance of the power spectrum of a general random field is explained in **Appendix A**, as well as a brief review of the Fourier Transform, though it is not essential for the understanding of our work.

Finally, the same way the code used to numerically solve a complex problem would not be part of this work, the intermediate steps towards the solutions of the evolution equations will be presented in **Appendix C**. While the equations do not have a straightforward resolution, the process bears no physical significance and would only make the text more difficult to read if presented in the main body.

This work is based on the ideas proposed by José Luis Sanz Estévez, who has been of invaluable help during its complete development. His are the concept of power perturbations γ_M derived from the density perturbations δ_M, and the idea of obtaining the expression of the evolution equation in terms of the scale factor a (see Chapter 4), from which he obtained preliminary versions of the complete solution equations for the perturbations.

CHAPTER 2

Theoretical Background

Contents

In this second chapter we will introduce the basic results from General Relativity and Cosmology relevant to this work, as well as our motivation in conducting the following chapters. Results found in any undergraduate level manual will mostly be given without proofs, while the more complex theory of metric perturbations and gauge-invariance will be treated in depth in Chapter 3.
The lecture notes from [**Baumann, 2016**] have served as a guide for this chapter, as well as Chapter 12 from [**Harwit, 2006**] for section 2.4, and [**Misner, 1973**] for some technical comments on General Relativity and further reading on this field.

2.1. Homogeneity and Isotropy

2.1.1. Cosmological Principle. The observations presented in the introductory chapter serve as evidence of the homogeneity and isotropy of the Universe surrounding the Earth. This induces the thought that our planet is not a particularly privileged point of the Universe, and that, from a different point similar observations (which themselves would correspond to past events, due to the finite nature of the speed of light) would be made. This homogeneity is only restricted to space, as evidence of the expansion of the Universe has been presented, which implies a different general state in the past.

These concepts are condensed in the so-called *Cosmological Principle*, which states that, for a fixed time, the properties of the Universe at large enough scales ($\gtrsim 1$ Mpc, see [**Ntelis, 2016**]) are the same for all observes in the Universe.

The problem of defining this particular "time" is still present, but can easily be solved by finding a particular property of the Universe that can be measured from any point and features the same value from every direction. A good example is the CMB (once dipole effects due to peculiar velocities have been subtracted), whose measured mean temperature is thought to decrease as the Universe expands. We can thus define a common *cosmic time* t for every point in the Universe.

Mathematically, this means that we can understand the Universe as a 4-dimensional manifold with the cosmic time t acting as a global coordinate, and where their t-constant or *spatial* hypersurfaces are isotropic and homogeneous for large scales. This geometric approach is fundamental in the definition of magnitudes and measurements, as well as in the derivation of the laws describing the evolution of the Universe. As a thorough approach to these concepts would require a complete text in General Relativity, which is not the aim of this work, we will only present the essential concepts and equations in the following sections. Some notes on the needed concepts of (pseudo-)Riemannian Geometry are presented in Appendix B.

One important postulate of this "geometrized approach" is the *Equivalence Principle*, appearing as direct consequence of the study of General Relativity over pseudo-Riemannian Manifolds, which states that general spacetime can be as accurately approximated as desired by a flat Minkowski spacetime given a sufficiently small neighborhood around the event to be studied.

2.1.2. FLRW Metric. In order to perform measures and study the Geometry and evolution of the Universe, it is necessary to define a pseudo-Riemannian metric (see Appendix B). By assuming the Cosmological Principle, we can suppose time and spatial parts of the metric tensor $g_{\mu\nu}$ can be separated orthogonally, so that it can be expressed as

$$g_{\mu\nu} = g_{\mu\nu}(t, \vec{x}) = \left[\begin{array}{c|c} -c^2 & 0 \\ \hline 0 & a^2(t)h_{ij}(\vec{x}) \end{array}\right], \tag{2.1.1}$$

where h_{ij} corresponds to the metric of a 3-dimensional space with constant Gaussian curvature K at the present time (K might vary along the evolution of the Universe, but remains constant for spatial hypersurfaces), and $a(t)$ is the *scale factor*, a positive valued function which serves as a way to parametrize the relative expansion of the Universe, with $a_0 = a(t_0) = 1$ at present time. The line element associated with $g_{\mu\nu}$ is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -c^2\mathrm{d}t^2 + a^2(t)h_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{2.1.2}$$

In order to simplify this expression, the cosmic time is replaced by the *conformal time* τ, related to t by

$$\mathrm{d}\tau := \frac{\mathrm{d}t}{a(t)}, \tag{2.1.3}$$

so that $g_{\mu\nu}$ can be now written as

$$g_{\mu\nu} = g_{\mu\nu}(\tau, \vec{x}) = a^2(\tau)\left[\begin{array}{c|c} -c^2 & 0 \\ \hline 0 & h_{ij}(\vec{x}) \end{array}\right]. \tag{2.1.4}$$

We will represent derivatives with respect to t with an apostrophe and to τ with a dot:

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \dot{f}(\tau), \qquad \frac{\mathrm{d}f}{\mathrm{d}t} = f'(t) = \frac{\dot{f}(t)}{a(t)} \tag{2.1.5}$$

As we will see in later chapters, most expressions in this work will use conformal time.
A popular expression of $g_{\mu\nu}$ in local coordinates is the *Friedmann-Lemaître-Robertson-Walker (FLRW) metric*, whose line element takes the following form in spherical coordinates:

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1-Kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right] \tag{2.1.6}$$

The Ehlers–Geren–Sachs theorem (see [**Ehlers, 1968**]) states that any isotropic and homogeneous universe has the FLRW as metric.

The distances, volumes and other metric elements obtained from the metric tensor h_{ij} are called *comoving*, as their value is independent of the expansion of the Universe by a. For example, from the comoving distance χ, the proper distance $D_P = a\chi$ can be obtained, which takes into account the expansion of the Universe, meaning that two objects will be increasingly apart from one another as the Universe expands, even if they do not move from their respective locations.
If we now obtain how the proper distance to a given object at a fixed comoving coordinate χ changes with the cosmic time t,

$$\frac{\mathrm{d}D_P}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[a(t)\chi\right] = \frac{a'(t)}{a(t)}D_P \tag{2.1.7}$$

Hubble's Law is obtained, which relates the proper velocity an object "drifts" from us to its proper distance D_P by means of a function dependent only of time, *Hubble's parameter*:

$$H(t) = \frac{a'(t)}{a(t)} \tag{2.1.8}$$

Using data from the *Planck Mission*, the Hubble parameter at the present takes a value of $H_0 = H(t_0) = 68.34 \pm 0.81\ \mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$ (see [**Planck Col.-Param., 2018**], page 27). As we foreshadowed earlier, conformal time will be of great importance along this work, so it is convenient to define the *conformal Hubble parameter*:

$$\mathcal{H}(\tau) := \frac{\dot{a}(\tau)}{a(\tau)} = a(\tau)H(\tau) \tag{2.1.9}$$

As $a_0 = 1$, it is immediate that $H_0 = \mathcal{H}_0$. The Hubble parameter H has inverse time units, being positive for expanding universes and negative for collapsing ones. While it cannot be assumed to have had a constant value throughout the complete history of the Universe, it can be used to obtain characteristic scales for the age of the universe, $t \sim H^{-1}$ (Hubble time), and the size of the observable Universe, $d \sim cH^{-1}$ (Hubble scale or *horizon*).

2.2. Stress-Energy Tensor. Equation of State

While the geometric properties of spacetime are contained in the metric tensor $g_{\mu\nu}$, the content in energy and matter is given by the *stress-energy tensor,* $T^{\mu\nu}$, a symmetric 2-contravariant tensor, which in General Relativity is defined as the flux of the μ-th component of the 4-momentum p^α through a x^ν-constant surface. For a perfect fluid with (energy-matter) density ρ and pressure P, the stress-energy tensor is found to have the following expression (read section 5.5 from [**Misner, 1973**]) :

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu + P g^{\mu\nu}, \tag{2.2.1}$$

where u^α is the 4-velocity of the fluid at each point. Considering a frame where the fluid in question is still, $u^\alpha = (c, 0, 0, 0)$, and using that locally we can work on a Minkowski space-time, with $g_{\mu\nu} = \eta_{\mu\nu} = \text{Diag}(-1, 1, 1, 1)$, we can express the mixed stress-energy tensor as

$$T^\mu_\nu = T^{\mu\alpha} g_{\nu\alpha} = \left(\rho + \frac{P}{c^2}\right) u^\mu u_\nu + P\delta^\mu_\nu \equiv \text{Diag}(-c^2\rho, P, P, P) \tag{2.2.2}$$

of which we will make use later.

Similar to the conservation of energy and momentum of Classical Mechanics, the stress-energy tensor is conserved locally (read sections 5.8 and 5.9 from [**Misner, 1973**]), which translates in that its divergence vanishes:

$$T^\alpha_{\beta;\alpha} = \nabla_\alpha T^\alpha_\beta = 0 \qquad \text{for} \quad \beta \in \{0, 1, 2, 3\}, \tag{2.2.3}$$

where ∇_α denotes the covariant derivative with respect to ∂_α, also expressed via the $_{;\alpha}$ subindex (see more on this in Appendix B). A brief calculation shows that in the FLRW metric, the following continuity equation is obtained from $T^\mu_{0;\mu} = 0$:

$$\dot{\rho} + 3\mathcal{H}\left(\rho + \frac{P}{c^2}\right) = 0 \tag{2.2.4}$$

2.2.1. Equation of State. Multiplying eq. 2.2.4 by a^3, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\rho a^3) = -\frac{3}{c^2} P\dot{a}a^2 \tag{2.2.5}$$

Now, dividing by $\dot{a}$ it is immediate that $\dot{a}^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}a}$, so

$$\frac{\mathrm{d}}{\mathrm{d}a}(\rho a^3) = -\frac{3}{c^2} P a^2 \tag{2.2.6}$$

It is easy to see that this last equation has an easy solution in the case $\rho \propto P$. This is the case for perfect fluids for which pressure and density are related via the *equation of state* ω, and an associated first-order quantity known as *sound velocity*, c_s

$$P = \omega c^2 \rho, \qquad c_s^2 := \frac{\partial P}{\partial \rho}, \tag{2.2.7}$$

which will appear multiple times along this work after making use of the chain rule. In general, for a perfect fluid with constant equation of state, $c_s^2 = c^2\omega$.
It is easy to see now that for a perfect fluid with constant equation of state ω, eq. 2.2.6 has as a solution

$$\rho(a) = \rho_0 \left(\frac{a}{a_0}\right)^{-3(\omega+1)} \tag{2.2.8}$$

Along this work we will work with three different equations of state:

- **Dust**, $\omega = \mathbf{0}$: We can think of the matter distribution at large scales, such as galaxies, as particles of dust, as their size is irrelevant compared to the extension of the Universe and the distance between one another, so the non-gravitational interaction between this "particles" (what we could interpret as pressure) is negligible, and thus $P_m = 0$. According to eq. 2.2.8, $\rho_m(a) \propto a^{-3}$, as the different dust particles become more isolated as the Universe expands.
- **Radiation**, $\omega = \frac{\mathbf{1}}{\mathbf{3}}$: From the thermodynamic equation of state of the photon gas ($U = 3PV$, with U and V being its total energy and volume) it is immediate to see that its pressure is proportional to its energy density by a $\frac{1}{3}$ factor, so that $P_\gamma = \frac{c^2}{3}\rho_\gamma$ (the c^2 term is included to keep the density in mass/volume units). In this case, $\rho_\gamma(a) \propto a^{-4}$.

- **Vacuum energy/Cosmological Constant** Λ, $\omega = -1$: While this concept will be introduced in the following section, we can consider now a Universe without any content of matter or radiation. By the Einstein Equations (which will be introduced in the next section too), the evolution of the metric is related to the stress-energy tensor $T^{\mu\nu}$. As the Universe would undergo evolution even when empty, in this case $T^{\mu\nu}$ cannot simply be a null tensor. Instead, it is sensible to suppose that the stress-energy tensor associated to the space-time geometry must be proportional to the metric, $T^{\mu\nu} \propto g^{\mu\nu}$. By comparison to eq. 2.2.1, it is easy to see that this is achieved for $P = -\rho c^2$. This corresponds to a $\omega = -1$ equation of state, corresponding to a fluid with some kind of "negative pressure". From eq. 2.2.8, we have that ρ_Λ is constant.

For a Universe filled with different fluids with their respective equations of state $\{\omega_i\}_i$, the stress-energy tensor $T^{\mu\nu}$ is the sum of the individual $(T_i^{\mu\nu})_i$. In the case each of them are perfect fluids, the Universe behaves as a perfect fluid itself, with pressure $\rho = \sum_i \rho_i$ and pressure $P = \sum_i P_i$, with each of the fluids evolving as $\propto a^{-3(\omega+1)}$.

2.3. Background Evolution. Λ-CDM model

In this section we will review the dynamics that govern the evolution of the Universe, as well as the equations describing its general behavior.

2.3.1. Einstein Equations. In the previous sections we have introduced the geometric and energy/material components of the Universe and described their evolution separately. The *Einstein Field Equations* are set of 10 partial differential equations (though the metric and stress-energy tensors have 16 components, due to their symmetric nature only 10 of them are independent[1]) which relates the geometry of spacetime, condensed in *Einstein Tensor* $G^{\mu\nu}$, directly obtainable from the metric $g_{\mu\nu}$, and the energy-matter content, given by the stress-energy tensor $T^{\mu\nu}$. The equations themselves, in the covariant form, take the following form:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu}} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.3.1}$$

where $R_{\mu\nu}$ corresponds to the *Ricci curvature tensor* and R to the *Ricci scalar* (more on this in Appendix B). The equation is found by equating a multiple of the stress-energy tensor to a divergence-free, symmetric combination of second order metric elements, while the $\frac{8\pi G}{c^4}$ factor is obtained by comparing to the Newtonian limit when trying to recover the Poisson's Equation for gravity ($\nabla^2\phi = 4\pi G\rho$, where ϕ is the gravitational potential).
The $\Lambda g_{\mu\nu}$ term appears as an integration constant in the derivation of the Einstein Equations, with the Λ factor called *Cosmological Constant*. From the discussion of the $\omega = -1$ equation of state in the previous section, it is easy to see that Λ can be included in the stress-energy tensor, inducing a vacuum energy term, with $\omega = -1$, and density $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$.

2.3.2. Friedmann-Lemaître Equations. Although eq. 2.3.1 is an elegant and brief expression, it hides complex mathematics such as second order derivatives and contraction of indexes. Assuming a FLRW metric and a (possibly multicomponent) perfect fluid Universe, after a lengthy but not too complicated calculation the following solutions, called *Friedmann Equations*, can be found, which we express in conformal time:

$$\dot{\mathcal{H}} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{a}}{a}\right) = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) a^2 = -\frac{4\pi G}{3}\rho(1+3\omega)a^2 \tag{2.3.2a}$$

$$\mathcal{H}^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho a^2 - K \tag{2.3.2b}$$

Analogous equations in cosmic time t can easily be found from these, but as we will work with conformal time τ we will not write them here in order not to confuse the reader. These, together with the continuity equation (eq. 2.2.4),

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = -3\mathcal{H}\rho(1+\omega) \tag{2.3.3}$$

can be used to obtain solutions for the Universe evolution. Obviously the use of ω in the previous equations is only valid for single component Universes.

[1]By using the four *Bianchi identities*, which we will not explain here, the independent components are reduced to 6. As this is not fundamental for this work, we will leave this as a mathematical curiosity (see Section 15.1 from [**Misner, 1973**]).

2.3.3. Components of the Universe. Λ-CDM model. While the Friedmann Equations describe the overall evolution of an isotropic, homogeneous Universe, other solutions different than the empty Universe ($\rho = 0$) ones requires from different fluids "filling" the Universe. The simplest, most mainstream model describing the different components of the Universe is the so-called Λ-*CDM model*, in which the following components exist:

- A *cosmological constant* Λ, associated to *dark energy* with negative pressure.
- *Cold dark matter*, that is a matter-like component (meaning that it can be described with a $\omega = 0$ equation of state), which is *non-baryonic* (not composed by ordinary matter such as electrons, protons and neutrons), has *non-relativistic velocities*, and does not interact with or emit electromagnetic radiation, being governed by gravitational (and possibly weak) force.
- *Ordinary matter* and *electromagnetic radiation*.

The Λ-CDM model, which is based on General Relativity, is able to give reasonable explanations of the main observed properties of the Universe, such as the CMB, the Large Scale Structure and the expansion of the Universe[2].

From equation 2.3.2b we find a critical density ρ_{crit} for which curvature K becomes null, serving as a "frontier" between positive and negative curvature universes. It expression it is easily found to be

$$\rho_{crit} = \frac{3\mathcal{H}^2}{8\pi G a^2} = \frac{3H^2}{8\pi G} \tag{2.3.4}$$

In the present time, its value is estimated as $\rho_{crit,0} = 8.5{\cdot}10^{-27}$ kgm^{-3} (see [**Planck Col.-Param., 2018**], page 27), approximately 5 atoms of hydrogen per cubic meter. We can use $\rho_{crit,0}$ as a normalization parameter in multicomponent universes, thus defining the *density parameter* (at present time), for a certain fluid x, as

$$\Omega_x := \frac{\rho_x(a_0)}{\rho_{crit,0}} \tag{2.3.5}$$

Their values have been measured experimentally by several missions, the latest of which was the *Planck Mission*. From [**Planck Col.-Param., 2018**], we have the following values[3] for the density parameters of matter, radiation and dark energy, respectively:

$$\Omega_m = 0.3111 \pm 0.0056, \quad \Omega_r = (9.19 \pm 0.17) \cdot 10^{-5}, \quad \Omega_\Lambda = 0.6889 \pm 0.0056 \tag{2.3.6}$$

The matter Ω_m parameter can be seen as the sum of the baryon and cold dark matter parameters, respectively $\Omega_b = 0.0493 \pm 0.0006$ and 0.265 ± 0.007, meaning that baryonic, "ordinary" matter only account for 15.8% of the matter of the Universe (and only 4.93% of the total mass-energy).

2.3.4. Scale factor determination for epoch limits. We will consider the Universe to be basically composed of dust-like matter (both dark and baryonic, with equation of state $\omega_m = 0$), electromagnetic radiation ($\omega_r = \frac{1}{3}$) and vacuum dark energy ($\omega_\Lambda = -1$), so that the density of the Universe at a certain scale factor a can be expressed as:

$$\rho(a) = \sum_x \rho_x(a) = \rho_m(a) + \rho_r(a) + \rho_\Lambda(a) \tag{2.3.7}$$

Other components, such as neutrinos, can be neglected unless performing very accurate calculations. Using eq. 2.2.8, the evolution of the total density of the Universe will change with a as

$$\rho(a) = \rho_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \rho_{r,0}\left(\frac{a}{a_0}\right)^{-4} + \rho_{\Lambda,0} \tag{2.3.8}$$

Using the density parameters $\{\Omega_x\}_x$ and the critical density at the present, we can then rewrite equation 2.3.8 as

$$\frac{\rho(a)}{\rho_{crit,0}} = \Omega_m\left(\frac{a}{a_0}\right)^{-3} + \Omega_r\left(\frac{a}{a_0}\right)^{-4} + \Omega_\Lambda \tag{2.3.9}$$

[2]There exist different experimental observations that challenge some aspects of the Λ-CDM model, as well as different extensions and alternatives to this model, but they are not the focus of this work.

[3]The actual value for Ω_r is not present at the Planck Mission 2018 results, but rather the redshift $z_{eq} = 3387 \pm 21$ at which the matter and radiation densities where equal, $\rho_m(z_{eq}) = \rho_r(z_{eq})$. As $z+1 = \frac{a_0}{a}$ and density evolves as given by eq. 2.2.8, including the density parameters we have that for z_{eq}, $\Omega_m(1+z_{eq})^3 = \Omega_r(1+z_{eq})^4$, so that $\Omega_r = \Omega_m(1+z_{eq})^{-1}$. The uncertainty of Ω_r has been found propagating those of Ω_m and z_{eq}.

While each of the components is present through the entire expansion of the Universe, as can be inferred from equation 2.3.9, the different exponents for a in each of the contributions imply that we can approximate the total density of the Universe as that of the dominating component in each "era". To find the a values at which this changes, we express the evolution of an arbitrary component using logarithms:

$$\frac{\rho_x(a)}{\rho_{crit}} = \Omega_x \left(\frac{a}{a_0}\right)^{-3(\omega_x+1)} \Leftrightarrow \log\left(\frac{\rho_x(a)}{\rho_{crit}}\right) = \log\Omega_x - 3(\omega_x+1)\log\left(\frac{a}{a_0}\right) \tag{2.3.10}$$

This way we find the a corresponding to each transition, assuming again $a_0 = 1$:

- **Matter dominated-Λ dominated**. We find a_m, for which $\rho_m(a_m) = \rho_\Lambda(a_m)$:

$$\log\Omega_m - 3\log\left(\frac{a}{a_0}\right) = \log\Omega_\Lambda \Rightarrow a_m = a_0\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}} = 0.767 \pm 0.004 \tag{2.3.11}$$

- **Radiation dominated-Matter dominated**. We find a_r, for which $\rho_r(a_r) = \rho_m(a_r)$:

$$\log\Omega_m - 3\log\left(\frac{a}{a_0}\right) = \log\Omega_r - 4\log\left(\frac{a}{a_0}\right) \Rightarrow a_r = a_0\frac{\Omega_m}{\Omega_r} = \frac{a_0}{1+z_{eq}} = (2.952 \pm 0.018)\cdot 10^{-4} \tag{2.3.12}$$

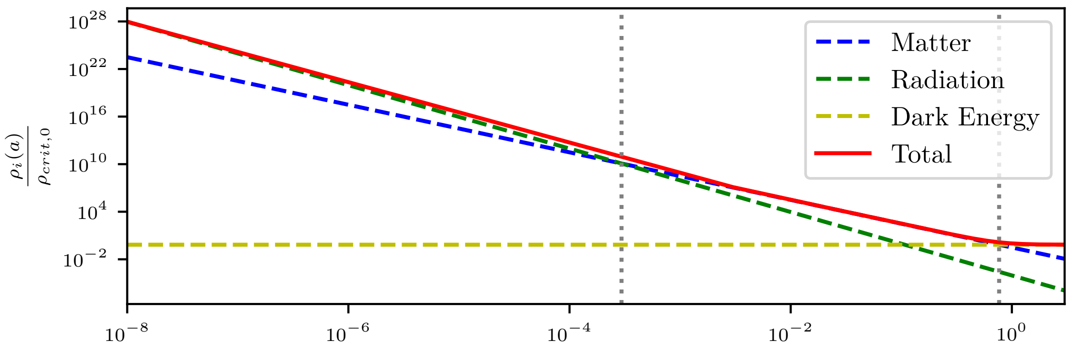

FIGURE 3. Comparison between the evolution of the total density ρ as a function of the scale factor $a(t)$ (which in turn evolves with time t), as well as the single component density evolution.

As it can be seen in Figure 3, the total density $\rho(a)$ can be approximated by single component expressions. The other two important scale factor delimiters, a_p and a_i (which, as we will see, correspond to the beginning and end of the inflation period, with a_r also acting as the initial scale factor of the Radiation dominated era), cannot be obtained from the FLRW metric and the derived dynamical equations (2.3.2), and will be obtained in section 2.4.

While this approximation slightly deviates from the true behavior of the Universe, specially in the "joining regions", it will allow us to consider the Universe in different eras, each governed by a different and single equation of state.

2.3.5. Evolution of the conformal Hubble parameter. In the previous subsection we carried out an important approximation, that is, that our multicomponent Universe can effectively be described by a series of eras in which one of the components dominated. For another important factor of our work, namely the evolution of the Hubble parameter, such drastic measures will not be needed.

Using eq. 2.3.2b and the density parameters it is easy to see that the conformal Hubble parameter evolves as

$$\mathcal{H}(a) = \mathcal{H}_0\sqrt{\Omega_m\left(\frac{a}{a_0}\right)^{-3} + \Omega_r\left(\frac{a}{a_0}\right)^{-4} + \Omega_\Lambda + \Omega_K\left(\frac{a}{a_0}\right)^{-2}}\left(\frac{a}{a_0}\right), \tag{2.3.13}$$

where Ω_K is an additional density parameter which takes into account the curvature contributions, being naturally defined as

$$\Omega_K = -\frac{c^2K}{\mathcal{H}_0^2} \tag{2.3.14}$$

In order to have $\mathcal{H}(a_0) = \mathcal{H}_0$, the following condition is imposed:

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_K = 1 \Longleftrightarrow \Omega_K = 1 - (\Omega_m + \Omega_r + \Omega_\Lambda), \tag{2.3.15}$$

meaning that the curvature of the Universe can be obtained bt accurately meassuring its components of matter, radiation and dark energy. Meassurements from the *Planck Mission* (read pages 40-41 from [**Planck Col.-Param., 2018**]) show curvature density parameter of $\Omega_k = 0.0007 \pm 0.0019$, meaning an approximately flat Universe, but does not rule out other possibilities.
It is immediate to find, imposing the maximum condition $\frac{\mathrm{d}}{\mathrm{d}a}\mathcal{H}(a) = 0$, that the conformal Hubble parameter reaches a minimum at $a_{min} = 0.6091$, where $\mathcal{H}_{min} = \mathcal{H}(a_{min}) = 0.8755\mathcal{H}_0 = 59.83$ kmMpc^{-1}s^{-1}. This value will serve us to obtain a threshold for the observable perturbations.

2.4. Arguments for inflation. Back of the envelope calculations for a_p, a_i and H_*

Despite that, as we mentioned at the beginning of this chapter, the Universe is isotropic and homogeneous at large scales, it nonetheless features inhomogeneities (otherwise the Universe would be composed by a perfect fluid, without any kind of structure forming, what would imply that our galaxy or even humanity would not exist) and anisotropies (though the Universe seems to be expanding in an isotropic way, individual galaxies and cosmic bodies have individual, peculiar movements) at small scales. Examples of these are the (small) inhomogeneities observed in the CMB, and the formation of galaxy clusters along the history of the Universe, as depicted in Figure 1.

As we shall see in the following chapters, these inhomogeneities are presented in the form of density perturbations $\delta\rho$ (respectively pressure perturbations δP) from the average $\bar{\rho}$ (resp. $\bar{P}$), which via the Einstein Equations result in curvature perturbations $\mathcal{R}$ in the FLRW spacetime (for more on these, read section 3.3 from chapter 3). These perturbations grow in magnitude due to gravitational effects along time, giving birth to large scale structure forming galaxies and galaxy clusters.
When looking for the origins of these observed inhomogeneities, we must look at the primordial Universe, when the energy/matter density was much higher than in the present, and small fractional fluctuations in this primordial fluid accounted for important absolute deviations from the mean values, and served as "cosmic seeds" for the formation and evolution of the inhomogeneities observed in later times. However, as we will see now, going back to primordial times is not exempt from problems.

The previous calculations to obtain the scale factors serving as limits for the different eras of the expansion of the Universe rely on the evolution of the scale factor driven by the densities of the different components, basically radiation, dust-like matter and dark energy. We will see that this assumptions do no suffice to explain the complete evolution of the Universe, being necessary the introduction of additional phenomena, such as inflation.
Using arguments regarding the area of Schwarzschild black holes and the Uncertainty Principle (see section 5.20 from [**Harwit, 2006**]) the most compact possible mass, m_P (*Planck mass*), with diameter l_p (*Planck length*), take the following values:

$$m_P := \sqrt{\frac{\hbar c}{G}} = 2.18 \cdot 10^8 \text{ kg}, \qquad l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.61 \cdot 10^{-35} \text{ m} \tag{2.4.1}$$

We can now define t_P, *Planck time*, as the time light would need to get from one side of the hypothetical length to the other:

$$t_P := \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} = 5.38 \cdot 10^{-44} \text{ s} \tag{2.4.2}$$

For shorter time intervals than this one, opposite sides of this hypothetical compact Planck length would not be causally connected, which would impede the thermal equilibrium between the different parts. Similarly, we can define a so-called *Planck density*,

$$\rho_P := \frac{m_P}{l_P^3} = \frac{c^5}{\hbar G^2} = 5.156 \cdot 10^{96} \text{ kgm}^{-3} = 6.07 \cdot 10^{122} \rho_{crit,0}, \tag{2.4.3}$$

having again that for greater densities the same problem as above.
As there are many evidences of the expansion of the Universe, we could wonder of the state of the Universe when all its components where concentrated in the smallest volume possible (that is, with a ρ_P density). As experimental data (read pages 40-41 from [**Planck Col.-Param., 2018**]) confirms our Universe is approximately flat, we can assume that its density is $\sim \rho_{crit,0} = 8.5 \cdot 10^{-27}$ kgm^{-3}. As the proper distance from Earth to the edge of the observable Universe is $d \approx 14.26$ Gpc$= 4.40 \cdot 10^{26}$ m, considering the observable Universe a sphere in an Euclidean space (that is, with volume $\frac{4}{3}\pi d^3$), its mass can be estimated as $M \sim 3 \cdot 10^{54}$ kg. Now, considering a ρ_P density, at a primitive time its volume would have been of about $9.7 \cdot 10^{-43}$ m^3, with a diameter of $6.14 \cdot 10^{-15}$ m. As light would take $t = 2.05 \cdot 10^{-23}$ s$\gg t_P$, we conclude that this parts would have been casually disconnected, leading to a similar reasoning

as in the previous paragraph.
This is specially relevant to the following reasoning. As we introduced at the beginning of this chapter, the Universe seems to be extremely isotropic, with small inhomogeneities and anisotropies. These could appear as relics of initial perturbations of pressure and density in the primordial Universe, that would later grow as time passed due to the effect of gravity[4]. The fact that observations find no differences in the anisotropies coming from different parts of the Universe, and as we have mentioned, the notion that initial mass composing the Universe at its beginning was causally disconnected is what it is called *horizon problem*.
There exist other problems that cannot be explained by Standard Cosmology in a FLRW Universe, such as the failure to detect several heavy subatomic particles, such as *magnetic monopoles*, hypothetical particles predicted to be produced at the early by some extensions of the Standard Model in particle physics, hot stages of the Universe, or the so called *flatness problem*. From eq. 2.3.13 it is easy to see that

$$1 - \frac{\rho(a)}{\rho_{crit}(a)} = -\frac{-c^2 K}{\mathcal{H}^2} \tag{2.4.4}$$

From eq. 2.3.13 it is inferred that $\mathcal{H}^{-1}$ grows with time, so $1 - \frac{\rho(a)}{\rho_{crit}(a)}$ would be unstable unless $K \sim 0$. As cosmological measurements imply $\rho(a_0) \sim \rho_{crit,0}$, our Universe must be almost flat, and with a "fine tuning" of $\rho \sim \rho_{crit}$ at the early Universe, which is at least surprising.

These problems motivated Alan Guth to propose the idea of *Cosmic Inflation* in 1981. According to his theory, at the beginning of the Universe, expansion of the superdense, ultrahot fluid composing it was slow enough to allow different regions to reach an equilibrium, thus achieving some kind of homogeneity. Along this expansion, the growth of the scale factor a make the density and temperature drop by several orders of magnitude, until $T \sim 10^{28}$ K, until matter and radiation no longer dominate, but rather *vaccum energy density* does. This temperature corresponds to a density[5] of $8.41 \cdot 10^{79}$ kgm^{-3}. Equating this to the vacuum density given by $\frac{\Lambda c^2}{8\pi G}$, we would find an *effective cosmological constant*[6] of

$$\Lambda_* = 1.57 \cdot 10^{54} \text{ m}^{-2}, \tag{2.4.5}$$

which is many orders of magnitude greater than the currently measured cosmological $\Lambda = 1.1056 \cdot 10^{-52}$ m^2. Considering a negative pressure, $\omega = -1$ equation of state, and a flat Universe $K = 0$ (which seems as an accurate approximation, as $K \propto a^{-2}$, making curvature negligible as the scale factor rapidly increases its value during inflation), the Continuity Equation 2.2.4 implies $\dot{\bar{\rho}} = 0$, so background density remains constant during inflation, and exponential growth occurs:

$$a(t) = a_p e^{H_* t}, \qquad \text{with } H_* = \sqrt{\frac{\Lambda_* c^2}{3}} \approx 2.17 \cdot 10^{35} \text{ s}^{-1} = 6.7 \cdot 10^{54} \text{ kms}^{-1}\text{Mpc}^{-1}, \tag{2.4.6}$$

where t is the cosmic time and a_p is the scale factor at the beginning of inflation. In order to have an idea of the magnitude of this exponential growth, we can compare H_* to $H_0 = 68.34$ kms^{-1}Mpc^{-1} (see [**Planck Col.-Param., 2018**], page 27), so that $H_* \sim 10^{53} H_0$.
We will now estimate the values of the scale factors at the beginning and end of inflation a_p and a_i. If we assume the observable Universe has a comoving diameter χ_0, then we would have

$$\left.\begin{array}{ccc} \chi_0 a_0 & \sim & c H_0^{-1} \\ \chi_0 a_p & \sim & c H_*^{-1} \end{array}\right\} \frac{a_p}{a_0} = \frac{H_0}{H_*} \Longrightarrow a_p \approx a_0 \frac{H_0}{H_*} \approx 1.02 \cdot 10^{-53} \tag{2.4.7}$$

Finally, to estimate a_i, we will use that, to explain the perceived homogeneity of the Universe, at least 60 *e-folds* (that is, a growth of at least an e^{60} factor during inflation) are required (read chapter 6 from [**Baumann, 2009**]). This way, we estimate a value of

$$a_i \approx e^{60} a_p = 1.142 \cdot 10^{26} a_p = 1.165 \cdot 10^{-27} \tag{2.4.8}$$

While, as we will introduce in the following subsection, during inflation most of the energy density is in the form of the inflaton potential $V(\phi)$ (which in a quite *naive* simplification we will assume to be constant, with its effects manifest through an effective Λ_*), this process ends when the potential decays in a steep manner, with the kinetic energy associated to the inflaton field increasing. Through damped-like oscillations, the inflation field, which is considered to be coupled with different fields from the Particle Standard Model fields, loses energy energy, which is in turn transferred to newly created particles, such

[4]One evidence that these perturbations grow with time is the fact that their scale at earlier times, such as the one when the CMB was produced, is much smaller than the observed inhomogeneities in the present LSS.

[5]Radiation energy density at a temperature T is given by $\rho_{rad} = aT^4$, with $a = 7.5658 \cdot 10^{-16}$ Jm^{-3}K^{-4}

[6]We are only using this Λ_* to easily model the exponential expansion during inflation, rather than giving it a physically-sensible meaning.

as baryons, photons, neutrinos, etc, in a process known as *reheating*. This new, hot, dense gas ends up (at least partly) reaching a thermal equilibrium with temperature T_{th} and density ρ_{th}, after which the evolution of the Universe can be described with the standard Friedmann Equations.
While radiation and density could exist before reheating, by this process their density grows immensely, which, as we will see in the following chapters, has an important effect in possible primordial density perturbations.

2.4.1. Initial fluctuations coming from Inflation[7]. The observed inhomogenieties of the Universe we observe in the present time, such as the galaxies and clusters, or the anisotropies of the CMB must have an origin in past times, and having evolved from simple perturbations in the energy-matter content of the Universe, or equivalently, via the Einstein Equation, in metric perturbations of spacetime. This way, it is natural to look for initial perturbations during the primitive stages of the Universe, where density was so high that small fractional perturbations $\frac{\rho(x)-\bar{\rho}(x)}{\bar{\rho}}$ would account for important differences. As a matter of fact, the same mechanism capable of explaining inflation allows for the existence of this initial perturbations, and how they can be used as "primordial seeds" for the growth of later perturbations.
Inflation can be explained via the existence of a primordial scalar field $\phi(t,\vec{x})$, called *inflaton field*, with an associated potential energy $V(\phi)$ and a stress-energy tensor $T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta} - V(\phi)\right)$. From the Euler-Lagrange (Field) Equations associated to ϕ, it is possible to obtain the associated "equations of motion". Following a reasoning analogous to the quantization of the harmonic oscillator, it is possible to express the observed possible values of the different Fourier modes $\hat{v}_k = a\hat{\phi}_k$ of the field, associated to different quantum states $|n\rangle$. While, as one would expect, the expected value for the field at the ground level is $\langle 0\,|\hat{v}_k|\,0\rangle = 0$, its variance $\left\langle 0\,\middle|\hat{v}_k^2\middle|\,0\right\rangle$ is not null, accounting for quantum fluctuations of this ground level. This quantum fluctuations in the inflaton field, via the Einstein Equations, give way to geometric perturbations, related in a simple manner to the so-called *curvature perturbations* $\mathcal{R}$ (as we will see in the following chapter, $\mathcal{R}$ accounts for perturbations of the (Ricci) scalar curvature):

$$\mathcal{R} = -\frac{\mathcal{H}}{\dot{\bar{\phi}}}\delta\phi, \qquad \langle|\mathcal{R}_k|\rangle = \left(\frac{\mathcal{H}}{\dot{\bar{\phi}}}\right)^2 \langle|\hat{\phi}_k^2|\rangle \tag{2.4.9}$$

These curvature perturbations behave as a random field, which have an associated power spectrum and dimensionless power spectrum (see Appendix A for more on this):

$$\langle\mathcal{R}(\vec{k})\mathcal{R}(\vec{k}')^*\rangle = \frac{1}{(2\pi)^{3/2}}\delta(\vec{k}-\vec{k}')P_{\mathcal{R}}(\vec{k}), \qquad \Delta^2_{\mathcal{R}}(\vec{k}) = \frac{k^3}{2\pi^2}P_{\mathcal{R}}(\vec{k}), \tag{2.4.10}$$

where $\mathcal{R}(\vec{k})$ is the Fourier transform of $\mathcal{R}(\vec{x})$. These coefficients contain the statistical information needed to obtain the relevant characteristics of these perturbations. The importance of these curvature perturbation relies in that for superhorizon scales, that is, for $k^{-1} \gg c\mathcal{H}^{-1}$, the quantum fluctuations $\Delta^2_{\mathcal{R}}(\vec{k})$ can be considered constant and, as we will see in the following chapter, for $\omega \neq -1$ $\mathcal{R}$ remain "frozen".

The idea is to evaluate the curvature fluctuations at superhorizon scales, where they will be conserved until horizon reentry (recall that $\mathcal{H}$ does not remain constant). It can be shown (read chapter 7 from [**Baumann, 2017**]) that, in this case, the spectrum associated to the curvature perturbations can be expressed near a certain k_* as

$$\Delta^2_{\mathcal{R}}(k) = A_s\left(\frac{k}{k_*}\right)^{n_s-1}, \qquad \text{where} \quad n_s - 1 := \left.\frac{\mathrm{d}\ln\Delta^2_{\mathcal{R}}}{\mathrm{d}\ln k}\right|_{k=k_*} \tag{2.4.11}$$

The n_s term is called *scalar spectral index*, and is used to quantify the scalar invariance of the fluctuations, with $n_s = 1$ corresponding to perfect invariance. The factor A_s immediately corresponds to $\Delta^2_{\mathcal{R}}(k_*)$. Experimental data from [**Planck Col.-Infl., 2018**] shows the following values:

$$A_s = \Delta^2_{\mathcal{R}}(k_*) = (2.445 \pm 0.096)\cdot 10^{-9}, \qquad \text{at} \quad k_* = 0.05\ \text{Mpc}^{-1} \tag{2.4.12}$$

and a scalar spectral index of $n_s = 0.9626 \pm 0.0057$, over the range $k \in [0.008, 0.1]$ Mpc^{-1}.

This "horizon evaluated" curvature perturbations are the standard candidates as the origin of the inhomogeneities and anisotropies observed in the later Universe, and we will use them as primordial seeds in our discussions. However, *other arbitrary initial perturbations which do not dissipate, but rather evolve*

[7]While the Physics behind inflation is of great beauty and theoretical interest, it is not within the goals of this work to give a detailed account of the inflationary concepts and derivations used. Due to length bounds, we are forced to merely enumerate the relevant results and recommend the reader the relevant text for the detailed derivations.

during inflation, or whose associated curvature perturbations do not freeze on superhorizon scales could be considered, serving as additional initial contributions.

CHAPTER 3

GR Linear Perturbations. Gauge Invariants.

Contents

In the second chapter we introduced the focus of this work: the density perturbations and their metric counterpart, the curvature perturbations. The definition of these magnitudes is not immediate, as their value can change or even cancel given a certain change of coordinates, as we will see. In this chapter we will define so-called *gauge-invariant* magnitudes, which do not change under general (linear) coordinate perturbations, and we will see how, combined with a perturbed version of the Einstein Equations, they can be used to find a *gauge-invariant evolution equation* for the density perturbations.

In order to simplify our calculations, *geometrized units* will be used, for which

$$c = 1, \qquad 8\pi G = 1 \tag{3.0.1}$$

Taking into account that c and $8\pi G$ have respectively $[LT^{-1}]$ and $[M^{-1}L^2T^{-2}]$ dimensions, we can easily recover "dimensionally sensible" expressions in case we want to obtain numerical values.
The results from this chapter, which is fundamental as a way to find the evolution equation we will use in the following chapter, are taken from [**Bardeen, 1980**], by J.M. Bardeen. While the results proven by the author were groundbreaking in the time they were published, the paper dodges many calculations and intermediate steps, and its written in a quite outdated notation. In order to completely understand metric perturbations and gauge-invariance, the complete derivation of the evolution equation is presented and updated, as well as given physical interpretation, of which the original article mostly lacked. Additional input was also taken from [**Dunsby, 1992**] and [**Baumann, 2017**].
This chapter makes extensive use of different concepts from Differential Geometry, which are reviewed in Appendix B.

3.1. Linear Metric Perturbations and Gauge Invariance

We will consider a FLRW background, with a background metric tensor $\overline{g}_{\mu\nu}(\tau, \vec{x})$ with spatial curvature K, filled with a perfect, isotropic fluid of mean density and pressure $\bar{\rho}(\tau)$ and $\bar{P}(\tau)$, with an associated stress-energy tensor $\overline{T}^{\mu}_{\nu}$, whose evolution is governed by the Friedmann and Continuity Equations (eqs. 2.3.2a, 2.3.2b and 2.2.4). The magnitudes $\bar{\rho}(\tau)$ and $\bar{P}(\tau)$ are related by the equation of state ω and the sound velocity c_s.
Considering now perturbations over this homogeneous, isotropic FLRW background, we can separate their time dependent part from the spacial one, the later being described by the *scalar Hemholtz Equation*, corresponding to the spatial part of the Wave Equation:

$$({}^{(3)}\nabla^2 + k^2)U = U_{;i}^{;i} + k^2 U = 0 \tag{3.1.1}$$

where ${}^{(3)}\nabla^2$ corresponds to the spatial Laplace-Beltrami operator[1] and k to the wave number, with $2\pi k^{-1}$ indicating the spatial scale of the perturbations with respect to the background comoving coordinates. Solutions U to eq. 3.1.1 are called *scalar harmonics*, which for curvature $K = 0$ correspond to plane waves. In our discussion, we will restrict ourselves, for simplicity's sake and following observational results[2] to the flat case ($K = 0$), but we will derive the evolution equation for a general curvature K in order to grasp the full meaning of the theory. We will use the scalar harmonics k modes as basis to express the perturbations, analogously to the role played by the different modes in a Fourier transform.

From a certain scalar harmonic U associated to a wavenumber k, vector and tensor harmonics can be defined:

$$V_i := -\frac{1}{k}U_{;i}, \qquad W_{ij} = \frac{1}{k^2}U_{;ij} + \frac{1}{3}h_{ij}U \tag{3.1.2}$$

[1]See Appendix A for this. We are using the ∇^2 symbol instead of Δ to avoid confussion with the power spectrum coefficients

[2]We again refer to pages 40-41 from [**Planck Col.-Param., 2018**]), showing a curvature density parameter of $\Omega_k = 0.0007 \pm 0.0019$.

Using the inverse matrix h^{ij} of the spatial metric, it is straightforward to show that W_{ij} is traceless:

$$\mathrm{Tr}\, W = W^i_i = h^{ij} W_{ij} = \frac{1}{k^2} \underbrace{h^{ij} U_{;ij}}_{U^{;i}_{;i}} + \frac{1}{3} \underbrace{h^{ij} h_{ij}}_{\delta^i_i = 3} U = -\frac{k^2}{k^2} U + \frac{3}{3} U = 0 \tag{3.1.3}$$

It can be shown that we can recover the scalar harmonics from the vector and tensor ones by means of divergences:

$$V^i_{;i} = kU, \qquad W^{ij}_{;ij} = \frac{2}{3}(k^2 - 3K)U \tag{3.1.4}$$

We can now define some general *linear metric perturbations* using these functions:

$$\begin{array}{rcl} g_{00} & = & -a^2(\tau)\,[1 + 2A(\tau)U(\vec{x})] \\ g_{0i} & = & -a^2(\tau) B(\tau) V_i(\vec{x}) \\ g_{ij} & = & a^2(\tau)\big\{[1 + \underbrace{2H_L(\tau)U(\vec{x})] h_{ij}(\vec{x})}_{\text{Longitudinal}} + \underbrace{2H_T(\tau) W_{ij}(\vec{x})}_{\text{Transversal}}\big\} \end{array} \tag{3.1.5}$$

where A, B, H_L and H_T are generic time dependent functions. This way $g_{\mu\nu}$ can be separated as

$$g_{\mu\nu} = \underbrace{a^2 \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & h_{ij} \end{array}\right]}_{\bar{g}_{\mu\nu}} + \underbrace{a^2 \left[\begin{array}{c|c} -2AU & -BV_i \\ \hline -BV_i^T & 2H_L U h_{ij} + 2H_T W_{ij} \end{array}\right]}_{\delta g_{\mu\nu}} \tag{3.1.6}$$

Under the *Newtonian gauge* $B = H_T = 0$ (read section 4.2.1 from [**Baumann, 2016**]), A can be interpreted as the gravitational potential causing a time dilation, and H_L as the local perturbation of the average scale factor. As we will see in later sections of this chapter, under certain conditions $A = -H_L$.

The *stress-energy tensor perturbations* can be expressed as

$$\begin{array}{rcl} T^0_0 & = & -\rho = -\bar{\rho}\,[1 + \delta(\tau)U(\vec{x})] \\ T^0_i & = & (\bar{\rho} + \bar{P})(v(\tau) - B(\tau))V_i(\vec{x}) \\ T^i_j & = & \bar{P}\big\{\underbrace{[1 + \pi_L(\tau)U(\vec{x})]\,\delta^i_j}_{\text{Isot. Pressure}} + \underbrace{\pi_T(\tau) W^i_j(\vec{x})}_{\text{Anisot. Stress}}\big\} \end{array} \tag{3.1.7}$$

Where again δ, π_L and π_T are generic functions of time. While δ corresponds[3] to the fractional perturbation of density ρ, the ij entries of the stress-energy momentum feature more complex perturbations, expressed as fractional perturbations of the background pressure, namely a longitudinal stress $\pi_L(\tau)U(\vec{x})\delta^i_j$ and a traceless, anisotropic stress $\pi_T(\tau)W^i_j(\vec{x})$. As we shall see later, the pressure and density perturbations need not to be related by either the equation of state ω or the sound speed c_s. We can define the *rest matter* frame as that in which the energy flux/momentum density is null, $T_{0i} = T_{i0} = 0$. We then define u^μ as the 4-velocity of this frame with respect to the coordinate frame, with $\vec{v}$ being the 3-velocity associated to u^μ.

We are now prepared to start dealing with *gauge transformations*. Due to the freedom we have when choosing the functions composing the perturbation of the metric tensor, $\delta g_{\mu\nu}$, as well as coordinate changes, it is possible that some physical magnitudes, such as the one composing the stress-energy tensor, can be rendered null under certain coordinate changes (it is straightforward that $T_{0i} = T_{i0} = 0$ if we choose $B = v$), meaning that these perturbation magnitudes are "not real", or "virtual", as their values might change from one reference system to another, and it is always possible to find a certain coordinate frame in which they are null. This means that we cannot directly work with these magnitudes, as they depend on the coordinate frame in which they are defined.
This way it is fundamental to look for combinations of magnitudes that are left invariant under general coordinate transformations. These magnitudes, called *gauge invariant*, will be the ones which will carry true physical meaning.
The theory of cosmological perturbations is a complex and still developing field. Unlike other fields of Physics, in which gauge transformations are performed via actions of well defined, specific groups such as rotations or Lorentz transformation, in this case we are dealing with general coordinate transformation through arbitrary functions. In hopes of maintaining a certain simplicity that allows the problem to be approachable, we will consider small enough perturbations such that they can be approximated as linear perturbations. This way, we will consider the following general coordinate changes:

$$\begin{array}{rcccl} \tau & \longmapsto & \widetilde{\tau} & = & \tau + T(\tau)U(\vec{x}) \\ x^i & \longmapsto & \widetilde{x}^i & = & x^i + L(\tau)V^i(\vec{x}) \end{array} \tag{3.1.8}$$

[3] Do not confuse the fractional density perturbation δ (a scalar function) with the Kronecker delta δ^i_j (a $(1,1)$-tensor).

where T and L are arbitrary functions of time small enough so only first order effects are relevant. With this in mind, we can express the effect of this coordinate change on the scale parameter a and the spatial metric h_{ij}:

$$a(\widetilde{\tau}) = a(\tau + T(\tau)U(\vec{x})) = a(\tau)\left[1 + \mathcal{H}(\tau)T(\tau)U(\vec{x})\right] + \mathcal{O}\left(T^2\right) \tag{3.1.9}$$

$$h_{ij}(\widetilde{\vec{x}}) = h_{ij}(\vec{x} + L(\tau)\vec{V}(\vec{x})) = h_{ij}(\vec{x})\left[1 + \frac{1}{h_{ij}}\frac{\partial h_{ij}(\vec{x})}{\partial x^l}(\vec{x})L(\tau)V^l(\vec{x})\right] + \mathcal{O}\left(L^2\right) \tag{3.1.10}$$

As a tensor, the spacetime metric $g_{\mu\nu}$ transforms the following way:

$$g_{\mu\nu}(x) = \frac{\partial\widetilde{x}^\alpha}{\partial x^\mu}\frac{\partial\widetilde{x}^\beta}{\partial x^\nu}\widetilde{g}_{\alpha\beta}(\widetilde{x}), \tag{3.1.11}$$

where the tilde in $\widetilde{g}_{\alpha\beta}$ means that the perturbation functions $\{A, B, H_L, H_T\}$ from eq. 3.1.5 are replaced by their transformed counterparts $\{\widetilde{A}, \widetilde{B}, \widetilde{H}_L, \widetilde{H}_T\}$. Using this, these metric perturbation amplitudes will change as follows:

$$\widetilde{A} = A - \dot{T} - \mathcal{H}T, \qquad \widetilde{B} = B + \dot{L} + kT, \qquad \widetilde{H}_L = H_L - \frac{k}{3}L - \mathcal{H}T, \qquad \widetilde{H}_T = H_T + kL \tag{3.1.12}$$

Proof. First of all, it is important to remark that, following perturbation analysis, the spatial dependence of the perturbed magnitude will be the same as that of harmonic functions generating the metric and stress-energy perturbations, meaning that U, V_i and W_{ij} are not to be perturbed, but rather act as the amplitude of the perturbations. Furthermore, as we will only be working with first order terms, we can consider $UV_i, V_iV_j, V_iW_{jk} \approx 0$. Because of this, we will also use (see eq. 3.1.10, in which derivatives of h_{ij} are multiplied by V^l) that up to first order, $h_{ij} \approx \widetilde{h}_{ij}$.
As they will be used extensively over this proof, we will compute the derivatives of the new coordinates with respect to the old ones:

$$\begin{gathered}\frac{\partial\widetilde{\tau}}{\partial\tau} = 1 + \dot{T}(\tau)U(\vec{x}) \qquad \frac{\partial\widetilde{x}^i}{\partial\tau} = \dot{L}(\tau)V^i(\vec{x}) \qquad \frac{\partial\widetilde{\tau}}{\partial x^i} = T(\tau)U_{;i}(\vec{x}) = -kT(\tau)V_i(\vec{x}) \\ \frac{\partial\widetilde{x}^j}{\partial x^i} = \delta_i^j + L(\tau)V^j_{;i}(\vec{x}) = \delta_i^j + L(\tau)\left[-\frac{1}{k}U^{;j}_{;i}(\vec{x})\right] = \delta_i^j + L(\tau)\left[-kW_{ij}(\vec{x}) + \frac{k}{3}h_{ij}U(\vec{x})\right]\end{gathered} \tag{3.1.13}$$

We will first study how g_{00} transforms to obtain the transformation in A. As $\frac{\partial\widetilde{x}^i}{\partial\tau}$ is already of first order, when multiplied with any other element of first order, it will discarded. This way, the only relevant term is the time-time one. Now,

$$\begin{aligned} g_{00}(x) &= -a^2(\tau)\left[1 + 2AU(\vec{x})\right] = \left(\frac{\partial\widetilde{\tau}}{\partial\tau}\right)^2\widetilde{g}_{00}(\widetilde{x}) = -\left(1 + \dot{T}(\tau)U(\vec{x})\right)^2 a^2(\tau + T(\tau)U(\vec{x}))\left[1 + 2\widetilde{A}U(\vec{x})\right] \\ &= -\left(1 + \dot{T}(\tau)U(\vec{x})\right)^2 a^2(\tau)\left[1 + \mathcal{H}(\tau)T(\tau)U(\vec{x})\right]^2\left[1 + 2\widetilde{A}U(\vec{x})\right] \\ &\approx -a^2(\tau)\left[1 + 2\mathcal{H}(\tau)T(\tau)U(\vec{x}) + 2\dot{T}(\tau)U(\vec{x}) + 2\widetilde{A}U(\vec{x})\right], \end{aligned} \tag{3.1.14}$$

where second order elements have been discarded. By inspection, we now have

$$\widetilde{A} = A - \dot{T}(\tau) - \mathcal{H}(\tau)T(\tau) \tag{3.1.15}$$

For B we will work with g_{0i} taking a similar approach:

$$\begin{aligned} g_{0i}(x) &= -a^2(\tau)BV_i(\vec{x}) = \frac{\partial\widetilde{x}^\alpha}{\partial\tau}\left[\frac{\partial\widetilde{\tau}}{\partial x^i}\widetilde{g}_{\alpha 0} + \frac{\partial\widetilde{x}^j}{\partial x^i}\widetilde{g}_{\alpha j}\right] \\ &\approx \left(1 + \dot{T}(\tau)U(\vec{x})\right)\left[-kT(\tau)V_i(\vec{x})\widetilde{g}_{00}(\widetilde{x}) + \delta_i^j\widetilde{g}_{0j}(\widetilde{x})\right] + \dot{L}(\tau)V^j(\vec{x})\left[-kT(\tau)V_i(\vec{x})\widetilde{g}_{j0}(\widetilde{x}) + \delta_i^k\widetilde{g}_{kj}(\widetilde{x})\right] \\ &\approx \left(1 + \dot{T}(\tau)U(\vec{x})\right)\widetilde{g}_{0i}(\widetilde{x}) - kT(\tau)V_i(\vec{x})\widetilde{g}_{00}(\widetilde{x}) + \dot{L}(\tau)V^j(\vec{x})\widetilde{g}_{ij}(\widetilde{x}) \\ &= a^2(\tau)\left[1 + \mathcal{H}(\tau)T(\tau)U(\vec{x})\right]^2\Big\{-\left(1 + \dot{T}(\tau)U(\vec{x})\right)\widetilde{B}V_i(\vec{x}) + kT(\tau)V_i(\vec{x})(1 + 2\widetilde{A}U(\vec{x})) + \\ &\quad + \dot{L}(\tau)V^j(\vec{x})\left[\widetilde{h}_{ij}\left(1 + 2\widetilde{H}_LU(\vec{x})\right) + 2\widetilde{H}_TW_{ij}(\vec{x})\right]\Big\} \\ &\approx a^2(\tau)\left[1 + \mathcal{H}(\tau)T(\tau)U(\vec{x})\right]^2\left[-\widetilde{B}V_i(\vec{x}) + kT(\tau)V_i(\vec{x}) + \dot{L}(\tau)V^j(\vec{x})h_{ij}\right] \\ &\approx -a^2(\tau)\left[-\widetilde{B} + kT(\tau) + \dot{L}(\tau)\right]V_i(\vec{x}), \end{aligned} \tag{3.1.16}$$

where we have only taken the δ_i^j term of $\frac{\partial\widetilde{x}^j}{\partial x^i}$, as the L factor will be canceled once multiplied by any other perturbative term (in this case $\widetilde{g}_{0i}$ and $\dot{L}(\tau)V_j$). We have furthermore used the h_{ij} metric tensor, which can be used in spatial hypersurfaces, such as the ones in which V_j has its domain, to raise and lower indexes, $V^j(\vec{x})h_{ij} = V_i(\vec{x})$. It is then easy to see that

$$\widetilde{B} = B + kT + \dot{L} \tag{3.1.17}$$

Finally, for H_L and H_T we will use g_{ij}:

$$\begin{aligned} g_{ij}(x) &= a^2(\tau)\left[h_{ij}(1+2H_L U(\vec{x})) + 2H_T W_{ij}(\vec{x}))\right] = \frac{\partial\widetilde{x}^\alpha}{\partial x^i}\frac{\partial\widetilde{x}^\beta}{\partial x^j}\widetilde{g}_{\alpha\beta}(\widetilde{x}) \\ &= k^2T^2(\tau)V_i(\vec{x})V_j(\vec{x})\widetilde{g}_{00}(\widetilde{x}) - 2kT(\tau)V_i(\vec{x})\left\{\delta_i^j + L(\tau)\left[-kW_{ij}(\vec{x}) + \frac{k}{3}h_{ij}U(\vec{x})\right]\right\}\widetilde{g}_{0j}(\widetilde{x}) + \\ &\quad + \left\{\delta_i^k + L(\tau)\left[-kW_{ik}(\vec{x}) + \frac{k}{3}\widetilde{h}_{ik}U(\vec{x})\right]\right\}\left\{\delta_j^l + L(\tau)\left[-kW_{jl}(\vec{x}) + \frac{k}{3}\widetilde{h}_{lj}U(\vec{x})\right]\right\}\widetilde{g}_{kl}(\widetilde{x}) \\ &\approx a^2(\tau)\left[1+\mathcal{H}(\tau)T(\tau)U(\vec{x})\right]^2\left[2L(\tau)\left(-kW_{ij}(\vec{x}) + \frac{k}{3}h_{ij}U(\vec{x})\right)\right]\left(h_{ij}(1+2\widetilde{H}_L U(\vec{x})) + 2\widetilde{H}_T W_{ij}(\vec{x})\right) \\ &\approx a^2(\tau)\left\{h_{ij}\left[1 + 2\left(\widetilde{H}_L + \mathcal{H}(\tau)T(\tau) + 2\frac{k}{3}L(\tau)\right)U(\vec{x})\right] + 2\left[\widetilde{H}_T - 2kL(\tau)\right]W_{ij}(\vec{x})\right\} \end{aligned} \tag{3.1.18}$$

where we have used that $UV_i, V_iV_j, V_iW_{jk} \approx 0$, as we are only considering terms up to first order, so the two addends of the second line can be taken as zero. It is now immediate that

$$\widetilde{H}_L = H_L - \frac{k}{3}L - \mathcal{H}T, \qquad \widetilde{H}_T = H_T + kL \tag{3.1.19}$$

□

It is immediate to see that in the limit $T, L \sim 0$ the perturbation amplitudes are left unchanged.
As it will prove useful shortly, we can also obtain how the matter 3-velocity changes with these new coordinates. Taking into account velocity is a vector magnitude and thus will be expressed in terms of V^i, we have

$$\widetilde{v}V^i = \frac{\mathrm{d}\widetilde{x}^i}{\mathrm{d}\widetilde{\tau}} = \frac{\mathrm{d}}{\mathrm{d}\widetilde{\tau}}\left(x + L(\tau)V^i(\vec{x})\right) \simeq \frac{\mathrm{d}}{\mathrm{d}\tau}\left(x + L(\tau)V^i(\vec{x})\right) = \frac{\mathrm{d}x^i}{\mathrm{d}\tau}\dot{L}(\tau)V^i(\vec{x}) = (v + \dot{L}(\tau))V^i \tag{3.1.20}$$

where we have considered that the magnitudes involved depend only on time τ and not the other functions while deriving with respect to $\widetilde{\tau} = \tau + T(\tau)U(\vec{x})$. This way,

$$v \longmapsto \widetilde{v} = v + \dot{L}, \tag{3.1.21}$$

as a spatial coordinate change dependent on time will surely affect the perceived magnitude of the velocity. Furthermore, the density perturbation function will change as follows:

$$\rho \longmapsto \widetilde{\rho} = \bar{\rho}(\widetilde{\tau})[1+\widetilde{\delta}U] \approx [\bar{\rho}(\tau) + \dot{\bar{\rho}}(\tau)TU][1+\widetilde{\delta}U] = \bar{\rho}(\tau)\left[1 + \left(\widetilde{\delta} + \frac{\dot{\bar{\rho}}}{\bar{\rho}}T\right)U\right] + \mathcal{O}(U^2) \tag{3.1.22}$$

Considering U to be small enough, we now have that, as density perturbations are a physical, measurable magnitude which should be independent of the coordinate gauge chosen, we have

$$\delta = \widetilde{\delta} + T\frac{\dot{\bar{\rho}}}{\bar{\rho}} = \widetilde{\delta} - 3\mathcal{H}(1+\omega)T \Leftrightarrow \widetilde{\delta} = \delta + 3\mathcal{H}(1+\omega)T \tag{3.1.23}$$

were we have used eq. 2.3.3 for the density evolution. This can be interpreted through the fact that the background density evolves with time, so a temporal coordinate shift will affect the perceived density perturbation. Following the exact same reasoning, we find that the longitudinal isotropic stress is changed by

$$\widetilde{\pi}_L = \pi_L - T\frac{\dot{\bar{P}}}{\bar{P}} = \pi_L + 3(1+\omega)\frac{c_s^2}{\omega}\mathcal{H}T \tag{3.1.24}$$

while it is easy to check that π_T is not affected by these gauge transformations.
As we have seen, it is easy to find coordinate changes for which some of these functions can be set to zero, which motivates the search of *gauge-invariant magnitudes*, which are left unaltered under this kind of coordinate changes. One of the simplest quantities we can define, using eqs. 3.1.21 and the H_T term of 3.1.12, is the following velocity

$$\boxed{v_s := v - \frac{1}{k}\dot{H}_T} \tag{3.1.25}$$

for which it is easy to see that is left invariant by gauge transformations. It can be shown (read [**Bardeen, 1980**], pages 21 and 22), that the shear tensor of the matter velocity field can be expressed as $\sigma_{ij} = -akv_sW_{ij}$, so that v_s is proportional to the transversal deformation induced by the metric

perturbations.
Of greater importance is defining gauge invariant density perturbations. We will define

$$\boxed{\delta_M := \delta + 3(1+\omega)\frac{1}{k}\mathcal{H}(v-B)} \tag{3.1.26}$$

A simple calculation shows us the δ_M value does not change from a gauge coordinate transformation;

$$\delta_M \mapsto \widetilde{\delta}_M = [\delta + 3(1+\omega)\mathcal{H}T] + 3(1+\omega)\frac{1}{k}\mathcal{H}(v + \dot{L} - B - \dot{L} - kT) = \delta_M \tag{3.1.27}$$

where we have used eqs. 3.1.23, 3.1.21 and the B term of 3.1.12. It is immediate to see that $\delta_M = \delta$ if $v = B$, that is, when $\vec{v} = B\vec{V}$, meaning that the worldlines of the matter frame are orthogonal to a $\tau =$ constant (spacelike) hypersurface, as the temporal and spatial parts are separated in the metric tensor, see Figure 4. Though there exist other different gauge-invariant expressions for the density perturbations (see [**Bardeen, 1980**], page 22 for more on this), we will use δ_M in our calculations, as they are the natural choice to work with gauge-invariant density perturbations from the matter point of view.
Inspecting equations 3.1.23 and 3.1.24, it is easy to see that δ and π_L change analogously, and that the following magnitude, called *entropy perturbation*, is gauge invariant:

$$\boxed{\eta := \pi_L - \frac{\partial \ln \bar{P}}{\partial \ln \bar{\rho}}\delta = \frac{1}{\omega}(\omega\pi_L - c_s^2\delta)}, \tag{3.1.28}$$

as the background density and pressure are left invariant under small, linear perturbations. This new magnitude, which has not straightforward relation with the "thermodynamical" entropy, corresponds with the difference between the fractional pressure perturbation (recall that from eq. 3.1.7, the isotropic part of the pressure perturbations is given by $1 + \pi_L(\tau)U(\vec{x})$), and that which we could expect coming from the density perturbations, δ, and the relation between the background density and pressure, $c_s^2 = \frac{\partial \bar{P}}{\partial \bar{\rho}}$.
To further explain the importance of this entropy perturbations, we shall define the so called *adiabatic fluctuation*, for which the local state of matter, described by a certain magnitude F, at a certain point $(\tau, \vec{x})$ of spacetime is that of background mean value $\bar{F}(\tau)$, at a slightly different moment, $\tau + \delta\tau(\mathbf{x})$,

$$\delta F(\tau, \vec{x}) := \bar{F}(\tau + \delta\tau(\vec{x})) - \bar{F}(\tau) \approx \dot{\bar{F}}(\tau)\delta\tau(\vec{x}), \tag{3.1.29}$$

with the particularity that the "time shift" $\delta\tau(\vec{x})$ is the same for all magnitudes. This way, for density and pressure,

$$\delta\tau(\vec{x}) = \frac{\delta P(\tau, \vec{x})}{\dot{\bar{P}}(\tau)} = \frac{\delta\rho(\tau, \vec{x})}{\dot{\bar{\rho}}(\tau)} \tag{3.1.30}$$

We can use this to show that in the adiabatic fluctuation case,

$$\eta = \pi_L - \frac{c_s^2}{\omega}\delta = \frac{\delta P}{\overline{P}} - \frac{c_s^2}{\omega}\frac{\delta\rho}{\bar{\rho}} = \left(\frac{\dot{\bar{P}}}{\overline{P}} - \frac{c_s^2}{\omega}\frac{\dot{\bar{\rho}}}{\bar{\rho}}\right)\delta\tau = 0, \tag{3.1.31}$$

where we have used that $\bar{P} = \omega\bar{\rho}$ and $\dot{\bar{P}} = \frac{\partial \bar{P}}{\partial \tau} = \frac{\partial \bar{\rho}}{\partial \tau}\frac{\partial \bar{P}}{\partial \bar{\rho}} = c_s^2\dot{\bar{\rho}}$. This way, adiabatic perturbations feature null entropy perturbations, which explains the name of this term (in classical thermodynamics, entropy remains constant in adiabatic, reversible processes). This fact will prove useful in the following section.

Now, regarding metric perturbations, as we have four functions perturbing the metric tensor (namely A, B, H_L and H_T) and two gauge functions for coordinate changes (T and L), we can build $4 - 2 = 2$ gauge-independent metric quantities from them, which we will call *Bardeen potentials*

$$\boxed{\Phi_A := A + \frac{1}{k}\dot{B} + \frac{1}{k}\mathcal{H}B - \frac{1}{k^2}\left(\ddot{H}_T + \mathcal{H}\dot{H}_T\right); \qquad \Phi_H := H_L + \frac{1}{3}H_T + \frac{1}{k}\mathcal{H}B - \frac{1}{k^2}\mathcal{H}\dot{H}_T} \tag{3.1.32}$$

Using eqs. 3.1.12, it is immediate to see this quantities are left invariant by a gauge transformations:

$$\Phi_A \mapsto \widetilde{\Phi}_A = \left(A - \dot{T} - \mathcal{H}T\right) + \frac{1}{k}\left(\dot{B} + \ddot{L} + k\dot{T}\right) + \frac{1}{k}\mathcal{H}\left(B + \dot{L} + kT\right) - \frac{1}{k^2}\left(\ddot{H}_T + k\ddot{L}\right) - \frac{1}{k^2}\mathcal{H}\left(\dot{H}_T + k\dot{L}\right) = \Phi_A$$

$$\Phi_H \mapsto \widetilde{\Phi}_H = \left(H_L - \frac{k}{3}L - \mathcal{H}T\right) + \frac{1}{3}\left(H_T + kL\right) + \frac{1}{k}\mathcal{H}\left(B + \dot{L} + kT\right) - \frac{1}{k^2}\mathcal{H}\left(\dot{H}_T + k\dot{L}\right) = \Phi_H$$

The physical meaning of Φ_A and Φ_H, and why we have addressed them as "potentials" will become more clear in the next section.
Apart from these two, other gauge-invariant magnitudes can be obtained by combining previous expressions in a different manner, in turn giving way to equivalent evolution equations. One important notion to take into account is that, while this general approach through arbitrary metric perturbations can be

used to derive evolution equations such as the one we will find in this chapter, the interpretation of any physical result requires a specific *gauge choice* (in order not to have extra metric perturbation functions), thus having an implicit *gauge freedom* in which to interpret our results. In this work, we will mainly use the *time-orthogonal* ($v = B = 0$) and *Newtonian* ($B = H_T = 0$), both of which allow us to separate the temporal and spatial parts of the metric. A sketch of the effect of the different terms of the metric perturbations $\delta g_{\mu\nu}$, as well as a representation of the density perturbations in the time-orthogonal ($v = B = 0$) gauge is represented on Figure 4.

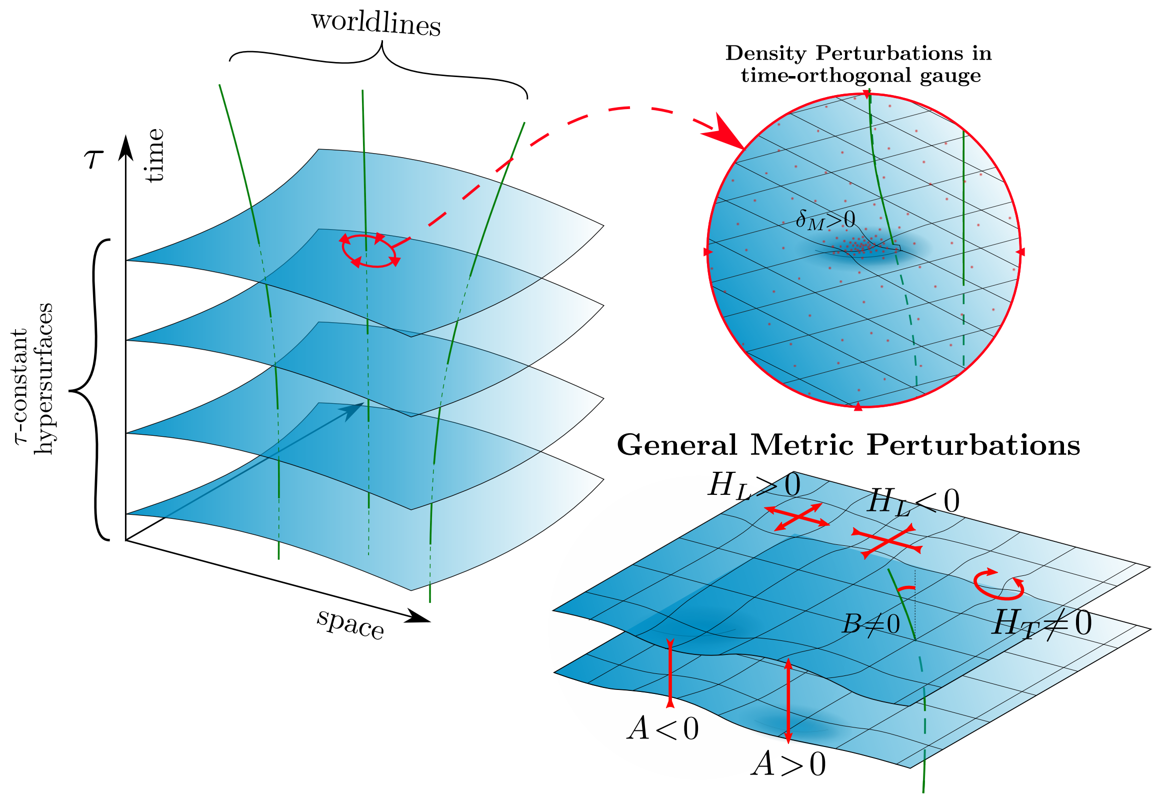

FIGURE 4. *Naive* representation of the metric perturbations $\delta g_{\mu\nu}$, and sketch of density perturbations in the time-orthogonal gauge, in which the worldlines are orthogonal to the τ-constant hypersurfaces.

3.2. Gauge Invariant Perturbation Evolution Equation

We are now able to find the relation between the energy-stress tensor perturbation and the metric perturbations, from the *Perturbed Einstein Field Equation*:

$$\delta G^{\alpha}_{\beta} = \delta T^{\alpha}_{\beta}, \tag{3.2.1}$$

where $\delta G^{\alpha}_{\beta} = \delta R^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}\delta R$ is the perturbed Einstein Tensor (recall we are working under the first order approximation). As derived in Appendix B, $\delta G^{\alpha}_{\beta}$ has the following components:

$$\delta G^0_0 = \frac{3}{a^2}\left\{-2\mathcal{H}\dot{H}_L + (\mathcal{H}^2 - \dot{\mathcal{H}})A - \frac{2k}{3}\mathcal{H}B - \frac{1}{3}(k^2 - 3K)\left(H_L + \frac{1}{3}H_T\right)\right\}U \tag{3.2.2a}$$

$$\delta G^0_i = \frac{2}{a^2}\left[-k\dot{H}_L - \frac{k}{3}\left(1 - \frac{3K}{k^2}\right)\dot{H}_T + k\mathcal{H}A - KB\right]V_i \tag{3.2.2b}$$

$$\begin{aligned}\delta G^i_j \;=\; & \frac{1}{a^2}\left\{\frac{1}{3}(k^2 - 3K)\left(H_L + \frac{1}{3}H_T\right) - 2\ddot{H}_L - \mathcal{H}(4\dot{H}_L - 2\dot{A}) + \left[\frac{-2k^2}{3} - (\mathcal{H}^2 - \dot{\mathcal{H}})\right]A + \frac{k}{3}\left(-2\dot{B} - 4\mathcal{H}B\right)\right\}\delta^i_j U \\ & + \frac{1}{a^2}\left[\ddot{H}_T + 2\mathcal{H}\dot{H}_T - k\left(\dot{B} + 2\mathcal{H}B\right) - k^2\left(H_L + \frac{1}{3}H_T + A\right)\right]W^i_j\end{aligned} \tag{3.2.2c}$$

Inspecting the perturbed Einstein tensor δG^{μ}_{ν}, the following two gauge-invariant combinations can be defined:

$$\delta G^0_0 - \frac{3}{k^2}\mathcal{H}(\delta G^0_i)^{;i} = -2\frac{k^2 - 3K}{a^2}\Phi_H U \tag{3.2.3a}$$

$$\delta G^i_j - \frac{1}{3}\delta^i_j \delta G^l_l = -\frac{k^2}{a^2}(\Phi_A + \Phi_H)W^i_j \tag{3.2.3b}$$

Using eq. 3.2.1, we equate δT^μ_ν with δG^μ_ν, so that eq. 3.2.3a transforms into

$$\delta T^0_0 - \frac{3}{k^2}\mathcal{H}(\delta T^0_i)^{;i} = -\bar{\rho}\delta U - \frac{3}{k^2}\mathcal{H}(1+\omega)\bar{\rho}(v-B)V^{;i}_i = -\bar{\rho}\delta_M U, \tag{3.2.4}$$

where we have used the definition of δ_M (see eq. 3.1.26) and that $V^{;i}_i = kU$, so that

$$\boxed{\bar{\rho}\delta_M = 2\frac{k^2 - 3K}{a^2}\Phi_H} \tag{3.2.5}$$

Comparing to the differential form of Gauss' law for gravity written as the Poisson's equation ($\nabla^2\phi = 4\pi G\rho$, where ϕ is the gravitational potential), and taking into account that in our units $8\pi G = 1$ and that $-k^2$ in the Fourier space corresponds to the Laplace-Beltrami operator in the real space, the function Φ_H can be seen as a perturbation of the gravitational field as a result of the gauge invariant density perturbation δ_M. Inspecting the way Φ_H is defined in equation 3.1.32, where H_L and H_T appear as first order terms, we deduce that from this equation the density perturbations δ_M translate in spatial, specifically curvature (for more on this, read page 22 from [**Bardeen, 1980**]), local perturbations, which in turn affect matter dynamics and evolution.

Similarly, for 3.2.3b, we have

$$-\frac{k^2}{a^2}(\Phi_A + \Phi_H)W^i_j = \delta T^i_j - \frac{1}{3}\delta^i_j\delta T^k_k = \bar{P}\left[\pi_L U\delta^i_j U + \pi_T W^i_j - \frac{1}{3}\delta^i_j\left(3\pi_L U + \pi_T W^i_i\right)\right] = \bar{P}\pi_T W^i_j \tag{3.2.6}$$

as W^i_j is traceless. This way,

$$\boxed{-\frac{k^2}{a^2}(\Phi_A + \Phi_H) = \bar{P}\pi_T} \tag{3.2.7}$$

This equation relates the Φ_H potential (which as we have stated, can be interpreted as a gravitational potential perturbation), and the Φ_A potential, which is in first term dominated by the metric temporal perturbation function A (other perturbation functions, namely B and H_T, appear accompanied by first or second order derivatives), via the anisotropic stress perturbations $\bar{P}\pi_T$. In the case we are dealing with anisotropic perturbations, this introduces a mismatch between the potentials regulating the temporal and spatial metric perturbations, which is in turn proportional to the square comoving reduced wavelength $2\pi\frac{a}{k}$ of each perturbation mode.

In the opposite case, if we are dealing with an isotropic fluid, $\pi_T \equiv 0$ and then

$$\Phi_A = -\Phi_H, \tag{3.2.8}$$

meaning that the temporal amplitude of the perturbations and the spatial curvature perturbations have equal magnitude and opposite sign. As the temporal metric perturbations (δg_{00}) appear with a negative sign (see eq. 3.1.5) this translates in $\delta g_{\mu\nu} \approx 2\Phi_H \bar{g}_{\mu\nu}$, if B and H_T are negligible (such as in the Newtonian Gauge case).

Apart from the Einstein's Equations, which related the metric and stress-energy perturbations, the local conservation of the stress-energy tensor ($T^\alpha_{\beta;\alpha} = 0$ for $\beta \in \{0,1,2,3\}$) can be used to obtain two additional "equations of motion", one for $\beta = 0$ and another for $\beta \in \{1,2,3\}$:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar{\rho}a^3\delta) + (\bar{\rho}+\bar{P})a^3(kv - kB + 3\dot{H}_L) + 3\bar{P}a^2\dot{a}\pi_L = 0} \tag{3.2.9a}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}(v-B) + \mathcal{H}(1-3c_s^2)(v-B) - kA - k\frac{\omega}{\omega+1}\pi_L + \frac{2}{3}k\left(1 - \frac{3K}{k^2}\right)\frac{\omega}{\omega+1}\pi_T = 0} \tag{3.2.9b}$$

where the first of the equations (*energy equation*) has been obtained[4] from $T^\alpha_{0,\alpha}$, and the second one (*momentum equation*[5]) from $T^\alpha_{i,\alpha}$. We will now try to express these equations in terms of gauge-invariant

[4] Bardeen expression in [**Bardeen, 1980**] for the energy equation has a mistake, as it does not feature the $-kB$ in the second term. However, under the gauge choice performed later this has no further effects.

[5] As it will be discussed in the following chapter, when working with a $\omega = -1$ equation of state, eq. 3.2.9b is not valid, with the expression obtained from $T^\alpha_{i;\alpha} = 0$ being

$$\pi_L = \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\pi_T \tag{3.2.10}$$

magnitudes. First of all, using equations 3.1.25 and 3.1.32,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}(v-B)+\mathcal{H}(1-3c_s^2)(v-B)-kA &= \frac{\mathrm{d}}{\mathrm{d}\tau}(v-B)+\mathcal{H}(1-3c_s^2)(v-B)+\dot{B}+\mathcal{H}B-\frac{1}{k}\left(\ddot{H}_T+\mathcal{H}\dot{H}_T\right)-k\Phi_A\\ &= \dot{v}_s+\mathcal{H}v_s-k\Phi_A-3c_s^2\mathcal{H}(v-B)\end{aligned} \tag{3.2.11}$$

Using this and equations 3.1.28 and 3.1.26, we can now manipulate eq. 3.2.9b:

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}(v-B)+\mathcal{H}(1-3c_s^2)(v-B)-kA-k\frac{\omega}{\omega+1}\pi_L+\frac{2}{3}k\left(1-\frac{3K}{k^2}\right)\frac{\omega}{\omega+1}\pi_T\\ &= \dot{v}_s+\mathcal{H}v_s-k\Phi_A-\underbrace{3c_s^2\mathcal{H}(v-B)}_{k(1+\omega)^{-1}c_s^2(\delta_M-\delta)}-k\frac{\omega}{\omega+1}\pi_L+\frac{2}{3}k\left(1-\frac{3K}{k^2}\right)\frac{\omega}{\omega+1}\pi_T\end{aligned} \tag{3.2.12}$$

so that it can now be written explicitly in a gauge-invariant form:

$$\boxed{\dot{v}_s+\mathcal{H}v_s=k\Phi_A+\frac{k}{1+\omega}\underbrace{\left(c_s^2\delta_M+\omega\eta\right)}_{\omega\pi_L\text{ if }\delta\equiv\delta_M}-\frac{2}{3}k\left(1-\frac{3K}{k^2}\right)\frac{\omega}{\omega+1}\pi_T} \tag{3.2.13}$$

This way, we can try to read eq. 3.2.13 as if the peculiar velocity v_s evolves in time ($\dot{v}_s$ being the acceleration) both as a consequence of the Universe expansion $\mathcal{H}v_s$ term (in the left side of the equation), and due to the perturbed gravitational potential $\sim\Phi_A$, the isotropic pressure perturbations ($\stackrel{\propto}{\sim}\pi_L$), and the anisotropic ones ($\propto\pi_T$).

Manipulation of eq. 3.2.9a is more complicated, as it features time derivatives in both the background and perturbation magnitudes. Trying to simplify things, we will work in the time-orthogonal gauge ($v=B=0$), in which the τ =constant spatial hypersurfaces are orthogonal to the matter worldlines (notice that in this case the metric tensor $g_{\mu\nu}$ has its time and space parts separated in two "boxes"). Using this gauge, eqs. 3.1.26 , 3.1.25 and 3.1.32 can be rewritten as

$$\delta_M=\delta,\qquad v_s=-\frac{1}{k}\dot{H}_T,\qquad \Phi_H=H_L+\frac{1}{3}H_T-\frac{1}{k^2}\mathcal{H}\dot{H}_T=H_L+\frac{1}{3}H_T+\frac{1}{k}\mathcal{H}v_s \tag{3.2.14}$$

Isolating H_L and deriving, we have

$$\begin{aligned}\dot{H}_L &= \dot{\Phi}_H-\frac{1}{3}\dot{H}_T-\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathcal{H}v_s\right)=\dot{\Phi}_H-\frac{1}{3}\dot{H}_T-\frac{1}{k}\left[\dot{\mathcal{H}}v_s+\mathcal{H}\dot{v}_s\right]\\ &= \dot{\Phi}_H+\frac{1}{3}kv_s+\frac{1}{6k}(\bar{\rho}+3\bar{P})a^2v_s-\frac{1}{k}\mathcal{H}\left[-\mathcal{H}v_s+k\Phi_A+\frac{k}{1+\omega}(c_s^2\delta_M+\omega\eta)-\frac{2}{3}k\left(1-\frac{3K}{k^2}\right)\frac{\omega}{\omega+1}\pi_T\right]\\ &= \dot{\Phi}_H+\frac{2}{3k}\left(k^2-3K+\frac{3}{2}(\bar{\rho}+\bar{P})a^2\right)v_s-\mathcal{H}\Phi_A-\frac{\dot{a}}{a}\frac{1}{1+\omega}(c_s^2\delta_M+\omega\eta)+\frac{2}{3}\mathcal{H}\left(1-\frac{3K}{k^2}\right)\frac{\omega}{1+\omega}\pi_T,\end{aligned} \tag{3.2.15}$$

where we have used eq. 3.2.13 to get rid of $\dot{v}_s$ and eqs. 2.3.2a and 2.3.2b to take account of the square and time derivative of $\mathcal{H}$. We now use the definition of η (eq. 3.1.28) and eq. 3.2.7 to express Φ_A in terms of Φ_H, and further regroup terms, to reach the following expression:

$$\dot{H}_L=\dot{\Phi}_H+\mathcal{H}\Phi_H+\frac{2}{3k}\left(k^2-3K+\frac{3}{2}(\bar{\rho}+\bar{P})a^2\right)v_s-\mathcal{H}\frac{\omega}{1+\omega}\pi_L+\frac{\bar{P}a^2}{k^2}\mathcal{H}\pi_T\left[1+\frac{2}{3}\frac{k^2-3K}{a^2(\bar{\rho}+\bar{P})}\right], \tag{3.2.16}$$

where the $\bar{P}=\omega\bar{\rho}$ relation has been used. Now, still looking at eq. 3.2.9a, and using equation 3.2.5, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar{\rho}a^3\delta_M)=2(k^2-3K)\frac{\mathrm{d}}{\mathrm{d}\tau}(a\Phi_H)=2(k^2-3K)(a\dot{\Phi}_H+\dot{a}\Phi_H)=2a(k^2-3K)\left(\dot{\Phi}_H+\mathcal{H}\Phi_H\right) \tag{3.2.17}$$

Applying eqs. 3.2.16 and 3.2.17 to eq. 3.2.9a, reorganizing the terms and canceling a common $2a[k^2-3K+\frac{3}{2}\left(\bar{\rho}+\bar{P}\right)]$ factor, we reach

$$\dot{\Phi}_H+\mathcal{H}\Phi_H=-\frac{1}{2}\frac{(\bar{\rho}+\bar{P})a^2}{k}v_s-\frac{\bar{P}a^2}{k^2}\mathcal{H}\pi_T \tag{3.2.18}$$

Although this equation is already gauge-invariant, we will further manipulate it by multiplying both sides by $2ka^2\left(1-\frac{3K}{k^2}\right)$ and using eq. 3.2.17 again, so we have

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\bar{\rho}a^3\delta_M\right)=-\left(1-\frac{3K}{k^2}\right)(\bar{\rho}+\bar{P})a^3kv_s-2\left(1-\frac{3K}{k^2}\right)\bar{P}a^2\dot{a}\pi_T} \tag{3.2.19}$$

We can read this equation as if the comoving density perturbations decrease due to the effect of the energy flux, given by the $\frac{k}{a}(\bar{\rho}+\bar{P})v_s$, as well as the anisotropic stress induced by $\bar{P}\pi_T$. For a more detailed account on this, we again refer to [**Bardeen, 1980**], page 22.

Finally we will consider the following. From Einstein's Equation we derived eqs. 3.2.5 and 3.2.7, which describe how the stress-energy perturbations and the Bardeen potentials Φ_A and Φ_H relate. From these, we were able to obtain a gauge-invariant formulation for the energy and momentum equations of motion (eqs. 3.2.19 and 3.2.13). However, these two equations, even if gauge-invariant, are still expressed in terms of both stress-energy variables and metric variables, namely v_s and Φ_A. To truly understand how physical (that is, not geometrical) perturbations evolve and affect one another, we will try to find a gauge invariant perturbation evolution equation completely written in terms of stress-energy variables. As eq. 3.2.13 is a second order equation (v_s itself is a first order magnitude, given the way it is defined), our equation will be of at least second order.

We will start by deriving eq. 3.2.19:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\bar\rho a^3\delta_M\right) = \left(1-\frac{3K}{k^2}\right)\Big[-\underbrace{(\dot{\bar\rho}+\dot{\bar P})}_{(1+c_s^2)\dot{\bar\rho}}a^3kv_s - (\bar\rho+\bar P)\frac{\mathrm{d}}{\mathrm{d}\tau}(a^3kv_s) - 2\dot a\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar Pa^2\pi_T) - 2\left(\dot{\mathcal{H}}+\mathcal{H}^2\right)(\bar Pa^3\pi_T)\Big], \tag{3.2.20}$$

where we have used that $\dot{\bar P} = \frac{\mathrm{d}\bar P}{\mathrm{d}\tau} = \frac{\mathrm{d}\bar P}{\mathrm{d}\bar\rho}\frac{\mathrm{d}\bar\rho}{\mathrm{d}\tau} = c_s^2\dot{\bar\rho}$. Now, using eq. 2.3.3 for $\dot{\bar\rho}$ and performing the derivation on a^3kv_s, we have that

$$\begin{aligned} -(1+c_s^2)\dot{\bar\rho}a^3kv_s - (\bar\rho+\bar P)\frac{\mathrm{d}}{\mathrm{d}\tau}(a^3kv_s) &= 3\frac{\dot a}{a}(1+c_s^2)(\bar\rho+\bar P)a^3kv_s - (\bar\rho+\bar P)a^3k\left(\dot v_s+3\mathcal{H}v_s\right) \\ &= (\bar\rho+\bar P)a^3k\left(3\mathcal{H}c_s^2v_s-\dot v_s\right) \end{aligned} \tag{3.2.21}$$

Using this and eqs. 2.3.2a and 2.3.2b to account for the $\dot{\mathcal{H}}$ and $\mathcal{H}^2$ terms, we can rewrite eq. 3.2.20 as

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\bar\rho a^3\delta_M\right) = \left(1-\frac{3K}{k^2}\right)\left[(\bar\rho+\bar P)a^3k\left(3\mathcal{H}c_s^2v_s-\dot v_s\right) - 2\dot a\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar Pa^2\pi_T) + \left(\frac{1}{3}(3\omega-1)\bar\rho a^2+2K\right)(\bar Pa^3\pi_T)\right] \tag{3.2.22}$$

Trying to cancel the v_s and $\dot v_s$ terms, we compute the following expression:

$$\begin{aligned} (1+3c_s^2)\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\bar\rho a^3\delta_M\right) &= \left(1-\frac{3K}{k^2}\right)\left[-(\bar\rho+\bar P)(1+3c_s^2)\mathcal{H}a^3kv_s - 2(1+3c_s^2)\mathcal{H}^2(\bar Pa^3\pi_T)\right] \\ &= \left(1-\frac{3K}{k^2}\right)\left[-(\bar\rho+\bar P)(1+3c_s^2)\mathcal{H}a^3kv_s - 2(1+3c_s^2)\left(\frac{1}{3}\bar\rho a^2-K\right)(\bar Pa^3\pi_T)\right], \end{aligned} \tag{3.2.23}$$

where we have used 2.3.2b for the $\mathcal{H}^2$ term. Now, adding up eqs. 3.2.22 and 3.2.23, we have

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\bar\rho a^3\delta_M\right) + (1+3c_s^2)\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\bar\rho a^3\delta_M\right) &= \left(1-\frac{3K}{k^2}\right)\Big[-(\bar\rho+\bar P)a^3k\left(\dot v_s+\mathcal{H}v_s\right) + \left(\omega-2c_s^2-1\right)(\bar\rho a^2)(\bar Pa^3\pi_T) \\ &\quad -2\dot a\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar Pa^2\pi_T) + 6c_s^2K(\bar Pa^3\pi_t)\Big] \end{aligned} \tag{3.2.24}$$

We can now use eq. 3.2.13 to cancel the v_s and $\dot v_s$ terms:

$$\begin{aligned} (\bar\rho+\bar P)a^3k\left(\dot v_s+\mathcal{H}v_s\right) &= (\bar\rho+\bar P)a^3k^2\Phi_A + \underbrace{\frac{\bar\rho+\bar P}{1+\omega}}_{\bar\rho}a^3k^2c_s^2\delta_M + \underbrace{\frac{\omega(\bar\rho+\bar P)}{1+\omega}}_{\bar P}a^3k^2\eta - \frac{2}{3}(k^2-3K)\underbrace{\frac{\omega(\bar\rho+\bar P)}{1+\omega}}_{\bar P}a^3\pi_T \\ &= -(1+\omega)(\bar\rho a^2)(\bar Pa^3\pi_T) - \frac{a^2}{2}(\bar\rho+\bar P)\frac{(\bar\rho a^3\delta_M)}{1-\frac{3K}{k^2}} + k^2c_s^2(\bar\rho a^3\delta_M) + k^2(\bar Pa^3\eta) \\ &\quad -\frac{2}{3}(k^2-3K)(\bar Pa^3\pi_T), \end{aligned} \tag{3.2.25}$$

where we have used eqs. 3.2.7 and 3.2.5 to write Φ_A and Φ_H in terms of stress-energy gauge-invariant magnitudes. Using eq. 3.2.25 to rearrange terms in eq. 3.2.24, we finally get to a gauge-invariant second order equation for the perturbation evolution expressed completely in stress-energy terms:

$$\boxed{\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(\bar\rho a^3\delta_M) + (1+3c_s^2)\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar\rho a^3\delta_M) + \left[(k^2-3K)c_s^2 - \frac{1}{2}(\bar\rho+\bar P)a^2\right](\bar\rho a^3\delta_M) \\ &= \left(1-\frac{3K}{k^2}\right)\left\{-k^2(\bar Pa^3\eta) + \frac{2}{3}\left[k^2+3(1+3c_s^2)K\right](\bar Pa^3\pi_T) + 2(\omega-c_s^2)(\bar\rho a^2)(\bar Pa^3\pi_T) - 2\dot a\frac{\mathrm{d}}{\mathrm{d}\tau}(Pa^2\pi_T)\right\} \end{aligned}} \tag{3.2.26}$$

As we shall see in the following chapter, though this equation does not incorporate the "complete" perturbed Einstein Equations (recall that only the derivative of $\delta G^0_i = \delta T^0_i$ is used, eq. 3.2.3a), not relevant information is lost, and the equation is valid for all scales. More on this will be introduced in the following section.

We shall now do a brief discussion of this evolution equation and its physical meaning. First of all, as we stated before obtaining eq. 3.2.26, the perturbation evolution is described by a second order relation, as it is obtained by operating with first and second order (specifically eq. 3.2.13) expressions.

Regarding the expression of the density perturbation evolution, we have introduced a so called *gauge-invariant comoving density perturbation*, $\varepsilon := \bar{\rho} a^3 \delta_M$, where the a^3 factor is introduced to take into account the volume expansion due to that of the Universe.
As for the structure of the scalar density perturbations evolution (eq. 3.2.26), we will first consider an homogeneous, adiabatic case, in which $\eta = \pi_T = 0$, that is, the right term of eq. 3.2.26 is null. Under this assumption, we have that the perturbed Einstein equations take the following form

$$\bar{\rho}\delta_M = 2\frac{k^2 - 3K}{a^2}\Phi_H, \qquad \Phi_A = -\Phi_H, \tag{3.2.27}$$

that is, the Bardeen potentials are one the opposite of the other and are directly proportional to the perturbation density. The motion equations, on the other hand,

$$\dot{v}_s + \mathcal{H}v_s = k\Phi_A + \frac{k^2 c_s^2}{1+\omega}\delta_M, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}(\bar{\rho}a^3\delta_M) = -\left(1 - \frac{3K}{k^2}\right)(\bar{\rho} + \bar{P})akv_s \tag{3.2.28}$$

It is immediate that in this case the velocity v_s evolution depends only on the density perturbation δ_M. Similarly, the gauge-invariant comoving density perturbation $\bar{\rho}a^3\delta_M$ evolves as a function of the background density (and pressure), and v_s. It is not difficult to combine these equations in a homogeneous expression featuring only $\bar{\rho}a^3\delta_M$.
In the non-homogeneous case, that is, with non-zero entropy and anisotropic stress perturbations, these terms serve as sources of for the perturbations, with the different factors accompanying them appearing as a result of the gauge-invariant character of the preceding equations. We assume that η and π_T are arbitrary functions of time, and as δ_M, η and π_T are respectively perturbations of scalar, vector and tensor character, their initial values can be considered to independent, the last two originating from a non-kinetic process, whose nature we will not consider in this work.
For a closed Universe ($K > 0$), there is the possibility that a $k = \sqrt{3K}$ perturbation mode exists, for which eq. 3.2.26 does not apply, as in its derivation we have repeatedly divided by $1 - \frac{3K}{k^2}$. This specific case will not be considered (as in the next chapter we will work in the $K = 0$ case), but discussion about it can be found in page 22 of [**Bardeen, 1980**].

Finally, following the reasons introduced in subsection 2.3.4, we have considered that the Universe is composed by a single fluid with equation of state ω and sound velocity c_s. If we considered a Universe with several components i, each with their respective stress-energy tensor gauge invariant perturbations $\{\delta_M^{(i)}, \eta^{(i)}, \pi_T^{(i)}\}_i$, which would be, related using the perturbed Einstein equations, to the metric perturbations, which are the same for every component. This would give way to a complicated system of coupled evolution equations, probably too convoluted to solve analytically, even under strong simplifications.

3.3. Curvature Perturbations and Horizon Crossing

When obtaining the perturbed Einstein's Equations ($\delta G^\mu_\nu = \delta T^\mu_\nu$), we respectively used the $(0,0)$ combined with the derivatives of the $(0,i)$ one, and (i,j) term to obtain eqs. 3.2.5 and 3.2.7. Instead of using the three different terms to obtain three independent equations, we used combinations of them to obtain two independent ones, thus leaving out of the equation information about the evolution of perturbations. As we shall see in the following chapter, solutions from eq. 3.2.26 will be valid at all scales, but to be able to corroborate this, we will obtain the evolution of perturbations at "superhorizon scales" (the meaning of this phrase will be clear shortly) along this section.
Before trying to put the "unused information" into use, we will consider the following. If we take τ-constant spacelike hypersurfaces and compute the spatial Riemann curvature tensor (see Appendix B for how R^i_j is defined, and section 3 from [**Bardeen, 1980**]), to first order we have

$$R^i_j = \frac{1}{a^2}\left[2K + \frac{4}{3}(k^2 - 3K)\left(H_L + \frac{1}{3}H_T\right)U\right]\delta^i_j - \frac{k^2}{a^2}\left(H_L + \frac{1}{3}H_T\right)W^i_j, \tag{3.3.1}$$

so that the Ricci curvature scalar looks like

$$R = R^i_i = \frac{1}{a^2}\left[6K + 4(k^2 - 3K)\left(H_L + \frac{1}{3}H_T\right)U\right] \tag{3.3.2}$$

The metric term $H_L + \frac{1}{3}H_T$ can be understood then as a "curvature perturbation". This expression is not gauge-invariant, though, for which we introduce the *(gauge-invariant) curvature perturbation*

$$\mathcal{R} := H_L + \frac{1}{3}H_T + \frac{1}{k}\mathcal{H}\,(B - v) \tag{3.3.3}$$

Note the similarity with Φ_H (eq. 3.1.32), obtained by changing the v term in $\mathcal{R}$ by $\frac{1}{k}\dot{H}_T$, with the two expressions coinciding for several gauge choices. We are now in conditions of computing the $\delta G^0_i = \delta T^0_i$ equation:

$$\frac{2}{a^2}\left[-k\dot{H}_L - \frac{k}{3}\left(1 - \frac{3K}{k^2}\right)\dot{H}_T + k\mathcal{H}A - KB\right] = (\bar{\rho} + \bar{P})(v - B) \tag{3.3.4}$$

We will now work in the so called *Newtonian Gauge*[6] (in which the gravitational perturbations coincide with those expected from Newtonian Gravitation), with $B, H_T = 0$, and we will furthermore consider a $\pi_T = 0$, perfect fluid. Under this assumptions, $A = \Phi_A = -\Phi_H$ and $\Phi_H = H_L$, so that eq. 3.3.4 can be written as

$$\dot{\Phi}_H + \frac{\dot{a}}{a}\Phi_H = -\frac{a^2}{2}(\bar{\rho} + \bar{P})\frac{v}{k} \tag{3.3.6}$$

As under this gauge choice $\mathcal{R} = \Phi_H - \mathcal{H}v$, we now have that for $\omega \neq -1$ (in which case $\bar{\rho} + \bar{P} = 0$),

$$\mathcal{R} = \Phi_H + \mathcal{H}\frac{2\left(\dot{\Phi}_H + \mathcal{H}\Phi_H\right)}{a^2(\bar{\rho} + \bar{P})} \tag{3.3.7}$$

Deriving eq. 3.3.7 with respect to τ and performing a series of tedious calculations involving eqs. 3.3.5a and 3.3.5c, which we will not reproduce here due to space constraints, but that can be followed in section 4.3.2 from [**Baumann, 2016**], we arrive to the following expression:

$$-\frac{a^2}{2}(\bar{\rho} + \bar{P})\dot{\mathcal{R}} = \frac{a^2}{2}\mathcal{H}\eta + \mathcal{H}c_s^2k^2\Phi_H \tag{3.3.8}$$

Working with adiabatic perturbations $\eta \equiv 0$, using that in Newtonian Gauge $\Phi_H = \mathcal{R} + \mathcal{H}k^{-1}v$, and that in a flat Universe (as the one we will work with in our calculations), eq. 2.3.2b looks like $\mathcal{H}^2 = \frac{1}{3}\bar{\rho}a^2$, we have

$$\frac{3}{2}\mathcal{H}^2(1 + \omega)\dot{\mathcal{R}} = \mathcal{H}c_s^2\mathcal{R}\left(1 + \frac{\mathcal{H}v}{k\mathcal{R}}\right) \tag{3.3.9}$$

Finally using that $\frac{\dot{\mathcal{R}}}{\mathcal{R}} = \frac{\mathrm{d}\ln\mathcal{R}}{\mathrm{d}\tau} = \mathcal{H}\frac{\mathrm{d}\ln\mathcal{R}}{\mathrm{d}\ln a}$ we have

$$\boxed{\frac{\mathrm{d}\ln\mathcal{R}}{\mathrm{d}\ln a} = \left(\frac{k}{\mathcal{H}}\right)^2\left[-\frac{2c_s^2}{3(1 + \omega)}\right]\left(1 + \frac{\mathcal{H}v}{k\mathcal{R}}\right)} \tag{3.3.10}$$

Assuming $\frac{\mathrm{d}\ln\mathcal{R}}{\mathrm{d}\ln a} \overset{\propto}{\sim} -\left(\frac{k}{\mathcal{H}}\right)^2$, we have that the evolution of $\mathcal{R}$ at superhorizon scales would be approximately of the form

$$\mathcal{R}(a) \sim \mathcal{R}_0 a^{-\left(\frac{k}{\mathcal{H}}\right)^2} \sim \mathcal{R}_0 \qquad \text{for} \quad \left(\frac{k}{\mathcal{H}}\right)^2 \ll 1 \tag{3.3.11}$$

This means that for perturbations with *super-Hubble scales* ($k \ll c^{-1}\mathcal{H}$) $\mathcal{R}$ will not evolve with a. As the Fourier mode of each perturbation remains constant over their evolution, but $\mathcal{H}$ changes as the Universe expands, the curvature perturbations will remain "frozen" once they cross the Hubble Horizon. Curvature perturbations are also conserved during inflation (whose exponential growth can be modelled through $\omega = -1$), but this is related to the way they are evaluated and obtained from quantum fluctuations, and not to eq. 3.3.10. For more on this read chapters 12 and 13 from [**Baumann, 2009**].

In order to see what happens to the density perturbations in super-horizon scales for a $\omega \neq -1$ evolution[7],

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[6] Under the Newtonian Gauge and $\pi_t = v = 0$ the perturbed Einstein equations greatly simplify to

$$\delta G^0_0 = \frac{3}{a^2}\left[-2\mathcal{H}\dot{\Phi}_H + (\mathcal{H}^2 - \dot{\mathcal{H}})\Phi_H - \frac{1}{3}(k^2 - 3K)\Phi_H\right]U = -\delta_M\bar{\rho}U = \delta T^0_0 \tag{3.3.5a}$$

$$\delta G^0_i = -\frac{2k}{a^2}(\dot{\Phi}_H + \mathcal{H}\Phi_H)V_i = 0 = \delta T^0_i \tag{3.3.5b}$$

$$\delta G^i_j = \frac{1}{a^2}\left[(k^2 - K)\Phi_H + (\mathcal{H}^2 - \dot{\mathcal{H}})\Phi_H - 6\mathcal{H}\dot{\Phi}_H - 2\ddot{\Phi}_H\right]\delta^i_jU = \bar{P}\pi_L\delta^i_jU = \delta T^i_j \tag{3.3.5c}$$

[7] As we indicated above, curvature perturbations coming are conserved in super-horizon scales during inflation. This is due to the fact that the variance of the quantum fluctuations that produce the $\mathcal{R}$ modes are constant for $k \ll c^{-1}\mathcal{H}$, which can be easily translated in the fact that the power spectrum $\Delta^2_{\mathcal{R}}(k)$ does not evolve in those scales. On the other hand, during the Λ Dominated Era the horizon is already shrinking again, so no new modes reenter the horizon, and the evolution of perturbations during this epoch does not need to be contemplated.

we must do as follows. We start from eq. 3.3.7, and use $\dot{\Phi}_H = \mathcal{H}\frac{\partial}{\partial \ln a}\Phi_H$ and eq. 2.3.2b to obtain

$$\mathcal{R} = \Phi_H + \frac{2}{3(1+\omega)}\left(\frac{\partial \Phi_H}{\partial \ln a} + \Phi_H\right) \tag{3.3.12}$$

Deriving once again with respect to $\ln a$, and using that in superhorizon scales $\frac{\partial}{\partial \ln a}\mathcal{R} \approx 0$, we have

$$\frac{\partial^2 \Phi_H}{\partial (\ln a)^2} + \frac{3\omega+5}{2}\frac{\partial \Phi_H}{\partial \ln a} = a^2\frac{\partial^2 \Phi_H}{\partial a^2} + \frac{3\omega+7}{2}a\frac{\partial \Phi_H}{\partial a} = 0 \tag{3.3.13}$$

This way the evolution for Φ_H outside the Hubble horizon has two modes:

$$\Phi_H(a) = \underbrace{Aa^{-\frac{3\omega+5}{2}}}_{\text{decaying}} + \underbrace{B}_{\text{constant}} \tag{3.3.14}$$

As $a^{-\frac{3\omega+5}{2}}$ decays quite rapidly for our equations of state $\omega \in \left(-1, \frac{1}{3}\right]$, we will only consider the constant modes. While our goal in this work is to obtain a *complete* evolution expression for the perturbations, in this section we are only trying to obtain the dominant evolution for super-horizon scales in order to check our results. In this case, as $\dot{\Phi}_H \approx 0$ for super-horizon scales, we can apply eq. 2.3.2b to eq. 3.3.7 to obtain

$$\mathcal{R} = \frac{3\omega+5}{3\omega+3}\Phi_H, \tag{3.3.15}$$

so, while both $\mathcal{R}$ and Φ_H are constant outside the horizon, the relationship between the two is not. It must be noted that when we obtained eq. 3.3.7 from eq. 3.3.6 we implicitly assumed $\omega \neq -1$. In the inflation/Λ-dominated case, eq. 3.3.6 takes the following form:

$$\dot{\Phi}_H + \mathcal{H}\Phi_H = 0 \Rightarrow \mathcal{H}\left(a\frac{\partial \Phi_H}{\partial a} + \Phi_H\right) = 0 \tag{3.3.16}$$

As $\mathcal{H} \neq 0$, the ODE has a solution the following expression:

$$\Phi_H(a) = \Phi_H(a_*)\frac{a_*}{a} \tag{3.3.17}$$

where a_* is any scale factor during the $\omega = -1$ epoch used as a reference. This way, Φ_H is not constant outside the horizon during this epoch. We can use our "Poisson Equation" (eq. 3.2.5) to obtain (for a flat $K = 0$ Universe and reincorporating fundamental constants)

$$\delta_M = \frac{\Phi_H}{4\pi G}\frac{k^2}{a^2\bar{\rho}(a)}, \tag{3.3.18}$$

which could be used to describe the evolution of the density perturbations outside the Hubble horizon.

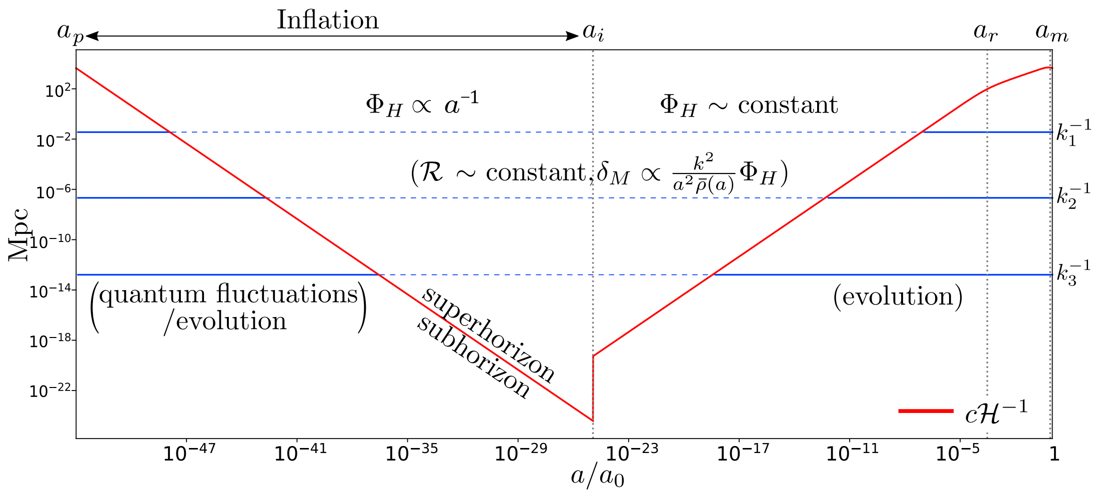

FIGURE 5. Diagram showing the relationship between the comoving Hubble horizon $c\mathcal{H}^{-1}$ and the different curvature perturbation modes. As $c\mathcal{H}^{-1}$ shrinks during inflation, the different curvature modes cross the horizon and "freeze" at different times/scale factors. After inflation the Hubble horizon grows again, and the different curvature perturbation modes reenter the horizon and start evolving again. The discontinuity in $c\mathcal{H}^{-1}$ at $a = a_i$ is due to the approximation that the cosmic Hubble parameter H_* remains constant during inflation.

CHAPTER 4

Evolution of perturbations

Contents

Using the evolution equation for gauge invariant comoving density perturbations (eq. 3.2.26) we deduced in the previous chapter, we will obtain solutions for different equations of state ω, describing different stages of the evolution of the Universe. In order to obtain analytic expressions, we will consider several simplifications, that will nonetheless result in interesting and realistic behaviors and phenomena. We will show how our solutions offer solutions for the perturbations evolution in terms of the initial conditions. Furthermore, for $\omega \neq -1$, our solutions will be valid for all scales, both inside (agreeing, but offering secondary corrections, with existent bibliography) and outside the horizon, in line with the results given in the previous chapter.
Along this chapter we will comment on the power spectrum solutions for each case and how they are affected by different initial conditions. We will finally study how the final states of each region serve as initial conditions for the following, and discuss the complete evolution of different perturbation Fourier modes along the Universe different eras.
As the resolution process of the different evolution equations can be quite long in some cases, and not really physically interesting, it is written in detail in Appendix C, so in this chapter we will start from the final solutions.
For comparison purposes with the solutions obtained from the standard formalism for Cosmological Perturbations we will use as references Chapter 5 from [**Baumann, 2016**] and Ch. 9 from [**Piattella, 2018**].

4.1. Basic Equation

Though along the previous chapter we followed various sources to discuss and give interpretation to the different gaugge invariant magnitude and their evolution, the evolution equation (eq. 3.2.26) was obtained following [**Bardeen, 1980**], by J.M. Bardeen, often considered one of the foundational texts in Cosmological Perturbation Theory. In that paper, Bardeen obtains gauge-invariant perturbation equations for scalar, vector and tensor perturbations (only the first of which we make use of), qualitatively discussing their analytic solutions. As the paper was published a year prior Guth's idea of cosmic inflation, the mechanisms through which initial perturbations could appear were not fully understood, and Bardeen focuses more on general solutions for the evolution equations than on how this initial perturbations evolve. Considering $K = \pi_T = \eta = 0$ conditions and a constant, arbitrary $\omega = c_s^2$ (which prevents further simplification as some evolution equation terms cancel under specific ω values, as we shall see), the behaviors of δ_M and $\xi = \frac{k}{\mathcal{H}} v_s$ (indicator of shear perturbations) are discussed in terms of $k\tau$. Under the same considerations as Bardeen, we will describe the evolution of the gauge-invariant fractional scalar density perturbations δ_M as a function of a (as we shall see this is a more natural choice), paying special attention to the initial perturbations driving evolution, and how these change along the different epochs of the Universe.

We will start recalling that the evolution equation (eq. 3.2.26) from the previous chapter gives the evolution of $\varepsilon = \bar{\rho}a^3\delta_M$ and, as we explained in the previous chapter, this gives the gauge invariant comoving density perturbations, where the fractional density perturbation δ_M gives account of of the over- and under-densities in τ-constant hypersurfaces, from the matter point of view, and where the a^3 takes account of the expansion of the comoving volume along that of the Universe.
As we introduced in the previous paragraphs , in hopes of simplifying our problem, we will assume ours is a perfect fluid (that is, $\pi_T \equiv 0$) and that its density perturbations are directly translated into longitudinal pressure perturbations by means of the equation of state ω and sound velocity c_s, so that the entropy perturbations are negligible, $\eta = 0$. This way, the right term of eq. 3.2.26 is canceled, leaving

$$\ddot{\varepsilon} + (1+3\omega)\frac{\dot{a}}{a}\dot{\varepsilon} - \left[\omega\left({}^{(3)}\nabla^2 + 3K\right) + \frac{1}{2}\underbrace{(\bar{\rho}+\bar{P})}_{(1+\omega)\bar{\rho}}a^2\right]\varepsilon = 0, \tag{4.1.1}$$

where we have replaced $-k^2$ by the spatial Laplace-Beltrami operator ${}^{(3)}\nabla^2$, thus returning to the direct space. This equation describes the time evolution of the perturbations. Because of the great scales involved in cosmology, time (be it t or τ) is not an easily measurable magnitude, often relying on other cosmological observables. On the other hand, the scale factor a is easily obtained from redshift measurements. Because of this, we will try to rewrite eq. 4.1.1 in terms of a, so our solutions are directly function of the expansion of the Universe rather than its age.
For mathematical convenience, and as a previous steps for later sections, where we will use distinct fluids, we have considered $\omega = c_s^2$. For the first derivation steps we will work with an arbitrary curvature K.

Going back to $\varepsilon = \bar{\rho}(a)a^3\delta_M$ and using that, under a dominant equation of state ω the background evolution $\bar{\rho}$ evolves as given by eq. 2.2.8, we have

$$\delta_M = \frac{\varepsilon}{\bar{\rho}(a)a^3} = \frac{1}{C}a^{3\omega}\varepsilon, \qquad \text{with} \quad C := \bar{\rho}_0 a_0^{3(\omega+1)} \tag{4.1.2}$$

These two expressions will prove useful in the next sections. We now use the expression for $\bar{\rho}(a)$ on eqs. 2.3.2a and 2.3.2b:

$$\dot{\mathcal{H}} = \frac{\partial}{\partial\tau}\left(\frac{\dot{a}}{a}\right) = -\frac{1+3\omega}{6}\bar{\rho}a^2 = -\frac{1+3\omega}{6}Ca^{-(1+3\omega)} \tag{4.1.3a}$$

$$\mathcal{H}^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{C}{3}a^{-3(1+\omega)} - K \tag{4.1.3b}$$

In order to use eq. 4.1.1, we will express the (conformal) time derivatives of ε in terms of a:

$$\dot{\varepsilon} = \frac{\partial\varepsilon}{\partial\tau} = \frac{\partial\varepsilon}{\partial a}\frac{\partial a}{\partial\tau} = \frac{\partial\varepsilon}{\partial a}\dot{a} \tag{4.1.4}$$

$$\begin{aligned}
\ddot{\varepsilon} &= \frac{\partial}{\partial\tau}\left(\frac{\partial\varepsilon}{\partial a}\dot{a}\right) = \frac{\partial}{\partial\tau}\left[\left(a\frac{\partial\varepsilon}{\partial a}\right)\frac{\dot{a}}{a}\right] = \left(\frac{\dot{a}}{a}\right)\frac{\partial}{\partial\tau}\left(a\frac{\partial\varepsilon}{\partial a}\right) + \left(a\frac{\partial\varepsilon}{\partial a}\right)\frac{\partial}{\partial\tau}\left(\frac{\dot{a}}{a}\right) \\
&= \left(\frac{\dot{a}}{a}\right)\frac{\partial}{\partial a}\left(a\frac{\partial\varepsilon}{\partial a}\right)\dot{a} + \left(a\frac{\partial\varepsilon}{\partial a}\right)\frac{\partial}{\partial\tau}\left(\frac{\dot{a}}{a}\right) = a\frac{\partial}{\partial a}\left(a\frac{\partial\varepsilon}{\partial a}\right)\left(\frac{\dot{a}}{a}\right)^2 + \left(a\frac{\partial\varepsilon}{\partial a}\right)\frac{\partial}{\partial\tau}\left(\frac{\dot{a}}{a}\right) \\
&= \frac{C}{3}a^{-3(1+\omega)}\left[a\frac{\partial}{\partial a}\left(a\frac{\partial\varepsilon}{\partial a}\right)\right] - \frac{1+3n}{6}Ca^{-(1+3\omega)}\left(a\frac{\partial\varepsilon}{\partial a}\right) - Ka\frac{\partial}{\partial a}\left(a\frac{\partial\varepsilon}{\partial a}\right) \\
&= \frac{C}{3}a^{-3(1+\omega)}\left(a\frac{\partial\varepsilon}{\partial a} + a^2\frac{\partial^2\varepsilon}{\partial a^2}\right) - \frac{1+3\omega}{6}Ca^{-(1+3\omega)}\left(a\frac{\partial\varepsilon}{\partial a}\right) - K\left(a\frac{\partial\varepsilon}{\partial a} + a^2\frac{\partial^2\varepsilon}{\partial a^2}\right)
\end{aligned} \tag{4.1.5}$$

where we have used eqs. 4.1.3. Regarding the second term of 4.1.1, we have

$$(1+3\omega)\frac{\dot{a}}{a}\dot{\varepsilon} = (1+3\omega)\left(\frac{\dot{a}}{a}\right)^2\frac{\partial\varepsilon}{\partial a}a = (1+3\omega)\left(a\frac{\partial\varepsilon}{\partial a}\right)\left(\frac{C}{3}a^{-3(1+\omega)} - K\right) \tag{4.1.6}$$

Applying these expressions to eq. 4.1.1 and multiplying by $\frac{3}{C}a^{1+3\omega}$, we have

$$a^2\frac{\partial^2\varepsilon}{\partial a^2} - \frac{3\omega}{C}a^{3\omega+1}\nabla^2\varepsilon + \frac{3(\omega+1)}{2}\left(a\frac{\partial\varepsilon}{\partial a} - \varepsilon\right) = \frac{3}{C}a^{3\omega+1}K\left[a^2\frac{\partial^2\varepsilon}{\partial a^2} + (3\omega+2)a\frac{\partial\varepsilon}{\partial a} - 3\omega\varepsilon\right] \tag{4.1.7}$$

As curvature evolves proportionally to a^{-2}, we will consider $K = \left(\frac{a}{a_0}\right)^{-2}K_0$, where K_0 is the current curvature of the Universe. This way,

$$\boxed{a^2\frac{\partial^2\varepsilon}{\partial a^2} - \frac{3\omega}{C}a^{3\omega+1}\nabla^2\varepsilon + \frac{3(\omega+1)}{2}\left(a\frac{\partial\varepsilon}{\partial a} - \varepsilon\right) = \frac{3a_0^2}{C}a^{3\omega-1}K_0\left[a^2\frac{\partial^2\varepsilon}{\partial a^2} + (3\omega+2)a\frac{\partial\varepsilon}{\partial a} - 3\omega\varepsilon\right]} \tag{4.1.8}$$

Data from the Planck mission (read pages 40-41 from [**Planck Col.-Param., 2018**]) show a curvature density parameter of $\Omega_k = -\frac{c^2 K}{\mathcal{H}^2} = 0.0007 \pm 0.0019$. As this value is small enough to allow a $K = 0$ flat Universe, and not large enough to unequivocally show an open or close Universe, we will consider the $K = 0$ case, which luckily simplifies eq. 4.1.8 to

$$\boxed{a^2 \frac{\partial^2 \varepsilon}{\partial a^2} - \frac{3\omega}{C} a^{3\omega+1} \nabla^2 \varepsilon + \frac{3(\omega+1)}{2} \left(a \frac{\partial \varepsilon}{\partial a} - \varepsilon \right) = 0} \tag{4.1.9}$$

Though eq. 4.1.9 is expressed in terms of ε, it is immediate that we can always recover the gauge-invariant fractional density perturbation δ_M:

$$\delta_M = \frac{\varepsilon}{\bar{\rho} a^3}, \qquad \text{with} \quad \bar{\rho} = \bar{\rho}_0 \left(\frac{a}{a_0} \right)^{-3(1+\omega)} \tag{4.1.10}$$

The equivalent to 4.1.9 in Fourier space is easily translated as

$$\boxed{a^2 \frac{\partial^2 \hat{\varepsilon}}{\partial a^2} + \frac{3\omega}{C} a^{3\omega+1} k^2 \hat{\varepsilon} + \frac{3(\omega+1)}{2} \left(a \frac{\partial \hat{\varepsilon}}{\partial a} - \hat{\varepsilon} \right) = 0}, \tag{4.1.11}$$

where $\hat{\varepsilon}(a, \vec{k}) = \mathcal{F}(\varepsilon(a))(\vec{k})$ is the Fourier transform of $\varepsilon(a, \vec{x})$.

Now we will use these expressions to obtain the evolution of the density perturbations with a for several equations of state ω in the following sections. Starting by some initial conditions at the start of inflation, we will use the perturbations at the end of each era as the initial values for the following. While one could naively assume that initial values for density perturbations δ_M are sufficient to fully describe the evolution in question, we have to take into account that eq. 3.2.26 (and thus eqs. 4.1.9 and 4.1.11) are second order equtions with respect to τ (resp. a), which means that two initial values (namely for δ_M and its derivative) are needed. This way, apart from δ_M, we will study the evolution of the dimensionless *power perturbations*:

$$\gamma_M := \frac{\partial \delta_M}{\partial \ln a} \tag{4.1.12}$$

Using eqs. 2.3.2b and 2.2.4 it is easy to see that, given a certain equation of state ω and a flat ($K = 0$) Universe, there is a direct relation between conformal time and the scale factor:

$$\left. \begin{array}{l} \rho \propto a^{-3(\omega+1)} \\ \dfrac{\dot{a}}{a} \propto \rho^{1/2} a \end{array} \right\} \dot{a} \propto a^{-\frac{3(1+\omega)}{2}} a^2 = a^{\frac{1-3+\omega)}{2}} \Rightarrow a^{\frac{1+3\omega}{2}} \propto \tau \Rightarrow a \propto \tau^{\frac{2}{1+3\omega}} \qquad \left(\omega \neq -\tfrac{1}{3}\right) \tag{4.1.13}$$

Notice that $\omega = -\frac{1}{3}$ does not correspond to any known fluid or phenomenon in Standard Cosmology. It is now easy to see that

$$\mathrm{d}a \propto \tau^{\frac{2}{1+3\omega}-1} \mathrm{d}\tau \Rightarrow \mathrm{d}\ln a \propto \mathrm{d}\ln \tau \Rightarrow \frac{\partial}{\partial \ln a} \propto \frac{\partial}{\partial \ln \tau} \tag{4.1.14}$$

It is immediate then that the power perturbations we have defined deriving δ_M with respect to $\ln a$ is proportional to those defined deriving with respect to $\ln \tau$, hence the name "power", as they describe the instantaneous change of δ_M with respect to a (or τ).
While we will obtain the evolution expressions for $\delta_M(a)$ and $\gamma_M(a)$, for statistical purposes (see Appendix A) it is more convenient to work with power coefficients associated to the perturbations involved:

$$\langle \hat{\delta}_M(a, \vec{k}) \hat{\delta}_M(a, \vec{k}')^* \rangle = \frac{1}{(2\pi)^{3/2}} \delta(\vec{k} - \vec{k}') P_{\delta_M}(a, \vec{k}), \qquad \Delta^2_{\delta_M}(a, \vec{k}) = \frac{k^3}{2\pi^2} P_{\delta_M}(a, \vec{k}), \tag{4.1.15a}$$

$$\langle \hat{\gamma}_M(a, \vec{k}) \hat{\gamma}_M(a, \vec{k}')^* \rangle = \frac{1}{(2\pi)^{3/2}} \delta(\vec{k} - \vec{k}') P_{\gamma_M}(a, \vec{k}), \qquad \Delta^2_{\gamma_M}(a, \vec{k}) = \frac{k^3}{2\pi^2} P_{\gamma_M}(a, \vec{k}), \tag{4.1.15b}$$

$$\langle \hat{\delta}_M(a, \vec{k}) \hat{\gamma}_M(a, \vec{k}')^* \rangle = \frac{1}{(2\pi)^{3/2}} \delta(\vec{k} - \vec{k}') \xi(a, \vec{k}), \qquad \Xi(a, \vec{k}) = \frac{k^3}{2\pi^2} \xi(a, \vec{k}), \tag{4.1.15c}$$

Here the $\xi(a, \vec{k})$ and $\Xi(a, \vec{k})$ functions account for the cross-correlation terms that appear in the power coefficients of δ_M and γ_M. As they are not power coefficients strictly speaking, we are not using the Δ^2_f notation. Unlike $\Delta^2_{\delta_M}$ and $\Delta^2_{\gamma_M}$, Ξ can take negative values, so it carries more information than just $\Delta^2_{\delta_M}$ and $\Delta^2_{\gamma_M}$. Apart from this, given the way power spectrum coefficients are defined, integrating for all k, in general $\Xi(a, \vec{k}) \neq \sqrt{\Delta^2_{\delta_M}(a, k) \Delta^2_{\gamma_M}(a, k)}$.

If we consider the density perturbations δ_M (an as a result the power perturbations γ_M) to be an homogeneous and isotropic random field (that is, the perturbations are expected not to be biased towards

any position or orientation), then its Fourier transforms will only depend on the modulus of the wave vector, $k = |\vec{k}|$. Because of this, we will express the perturbations and their power coefficients in terms of k, instead of $\vec{k}$.
Regarding the evolution of perturbations outside the horizon, in the previous chapter we showed that for $\omega \neq -1$ both $\mathcal{R}$ and Φ_H remained constant, while the density perturbations behaved as

$$\hat{\delta}_M = \frac{\Phi_H}{4\pi G}\frac{k^2}{a^2\bar{\rho}(a)} \tag{4.1.16}$$

As $\hat{\delta}_M$ is not constant, the power perturbation $\hat{\gamma}_M$ will be non-null, and actually related to $\hat{\delta}_M$ in a very simple way. From the definition, for $\omega \neq -1$, as Φ_H is constant,

$$\hat{\gamma}_M = \frac{\partial\hat{\delta}_M}{\partial \ln a} = \frac{\Phi_H}{4\pi G}k^2 a\frac{\partial}{\partial a}\left[a^2\bar{\rho}(a)\right]^{-1} = -\frac{\Phi_H}{4\pi G}k^2\frac{2a^2\bar{\rho}(a) + a^3\frac{\partial}{\partial a}\bar{\rho}(a)}{(a^2\bar{\rho}(a))^2} = -\frac{\Phi_H}{4\pi G}\frac{k^2}{a^2\bar{\rho}(a)}\left[2 + \frac{\partial\ln(\bar{\rho}(a))}{\partial \ln a}\right] \tag{4.1.17}$$

Regarding the logarithmic derivative of $\bar{\rho}(a)$, we must simply use eq. 2.3.3:

$$\frac{\partial\ln(\bar{\rho}(a))}{\partial\ln a} = \frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}(a)}{\partial\ln a} = \frac{1}{\mathcal{H}\bar{\rho}}\frac{\partial\bar{\rho}(a)}{\partial\tau} = \frac{\dot{\bar{\rho}}}{\mathcal{H}\bar{\rho}} = \frac{3\mathcal{H}(1+\omega)\bar{\rho}}{\mathcal{H}\bar{\rho}} = -3(1+\omega), \tag{4.1.18}$$

so that

$$\hat{\gamma}_M = \frac{\Phi_H}{4\pi G}\frac{k^2}{a^2\bar{\rho}(a)}(1+3\omega) = (1+3\omega)\hat{\delta}_M \tag{4.1.19}$$

As for the power coefficients of the perturbations outside the horizon, it is immediate that for $\omega \neq -1$

$$\Delta^2_{\gamma_M} = (1+3\omega)^2\Delta^2_{\delta_M}, \qquad \Xi = (1+3\omega)\Delta^2_{\delta_M} \tag{4.1.20}$$

We will use these expressions in order to obtain the initial values of the power coefficients $\Delta^2_{\gamma_M}$ and Ξ from $\Delta^2_{\delta_M}$, which, as we have seen, can be directly obtained from $\Delta^2_{\mathcal{R}}$, which is the only experimental measure we have related to the density. While this seems an important simplifications, as we will see, for superhorizon scales our complete expressions for the perturbation evolution will agree with eq. 4.1.19.

We will present the solutions for each era and comment their behavior for different initial conditions in sections 4.2 to 4.3. After a small detour for considering the $\omega = -1$ case, the complete evolution of the perturbations for specific initial values will be given in section 4.5.

From the calculated scale factor that serve as limits for the different epochs we obtained from measured the density parameters $\{\Omega_m, \Omega_r, \Omega_\Lambda\}$ ([**Planck Col.-Param., 2018**]), we will apply eq. 2.3.13 to obtain the limit k values for the modes that reenter the horizon at each epoch.

An important fact to notice is that, while the easiest way to interpret the perturbation evolution is by means of the expressions for $\hat{\delta}_M$ and $\hat{\gamma}_M$, if we wanted to use or compare to experimental data, as inhomogeneities and isotropies are an statistical concept, are measured as such, by means of the power spectrum coefficients. It is not easy to obtain $\hat{\delta}_M$ and $\hat{\gamma}_M$ from $\Delta^2_{\delta_M}$, $\Delta^2_{\gamma_M}$ and Ξ, as the power spectrum coefficients are obtained integrating the measured anisotropies all over the sky. Because of this, along this chapter we will plot and mainly comment these functions, as are the ones which could be compared with experimental data.

4.2. Epoch I: Radiation Dominated Era ($\omega = \frac{1}{3}$)

In the Radiation Era, dominated by ultrarelativistic particles (primary photons) and ranging from $a_i = 1.165 \cdot 10^{-27}$ to $a_r = 2.952 \cdot 10^{-4}$, the Universe can be considered to be governed by a $\omega = \frac{1}{3}$ equation of state, with eq. 4.1.9 taking the following form in the real space, a hyperbolic[1] PDE ($+4C^{-1}a^4 > 0$):

$$a^2\frac{\partial^2\varepsilon}{\partial a^2} + 2\left(a\frac{\partial\varepsilon}{\partial a} - \varepsilon\right) - \frac{1}{C}a^2\nabla^2\varepsilon = 0, \qquad \text{with} \quad C = 3\Omega_r H_0^2 a_0^2 \tag{4.2.1}$$

and in the Fourier space,

$$a^2\frac{\partial^2\hat{\varepsilon}}{\partial a^2} + 2\left(a\frac{\partial\hat{\varepsilon}}{\partial a} - \hat{\varepsilon}\right) + \frac{1}{C}a^2k^2\hat{\varepsilon} = 0 \tag{4.2.2}$$

[1]Recall that given an arbitrary second order PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Gu_y + Fu = v$$

we say that the PDE is *elliptic* if $B^2 - 4AC < 0$, *parabolic* if $A^2 - 4AC = 0$, or *hyperbolic* if $B^2 - 4AC > 0$.

As we will see, by comparing with eqs. 4.3.1 and 4.4.1 (evolution equations for matter dominated and inflationary/Λ-dominated eras, respectively) it is easy to see that eq. 4.2.2 features the most complicated form of the three, and its solutions are likely to have a more complicated (and thus interesting) form than the others. Equation 4.2.2 can be found to have as perturbation solutions

$$\hat{\delta}_M(z,k) = \left[\alpha(z)\hat{\delta}_{M,i}(k) + z_i^{-1}\beta(z)\hat{\gamma}_{M,i}(k)\right]\sin(z-z_i) + \left[\lambda(z)\hat{\delta}_{M,i*}(k) + z_i^{-1}\gamma(z)\hat{\gamma}_{M,i}(k)\right]\cos(z-z_i) \tag{4.2.3a}$$

$$\hat{\gamma}_M(z,k) = z\left\{\left[a(z)\hat{\delta}_{M,i}(k) + z_i^{-1}b(z)\hat{\gamma}_{M,i}(k)\right]\cos(z-z_i) + \left[c(z)\hat{\delta}_{M,i}(k) + z_i^{-1}d(z)\hat{\gamma}_{M,i}(k)\right]\sin(z-z_i)\right\} \tag{4.2.3b}$$

with the following power coefficients associated:

$$\begin{aligned}
\Delta^2_{\delta_M}(z,k) &= [\alpha(z)\sin(z-z_i) + \lambda(z)\cos(z-z_i)]^2\,\Delta^2_{\delta_M,i}(k)\\
&+ z_i^{-2}[\beta(z)\sin(z-z_i) + \gamma(z)\cos(z-z_i)]^2\,\Delta^2_{\gamma_M,i}(k)\\
&- 2z_i^{-1}[\alpha(z)\sin(z-z_i) + \lambda(z)\cos(z-z_i)]\,[\beta(z)\sin(z-z_i) + \gamma(z)\cos(z-z_i)]\,\Xi_i(k)
\end{aligned} \tag{4.2.4a}$$

$$\begin{aligned}
\Delta^2_{\gamma_M}(z,k) &= z^2\Big\{[a(z)\cos(z-z_i) + c(z)\sin(z-z_i)]^2\,\Delta^2_{\delta_M,i}(k)\\
&+ z_i^{-2}[b(z)\cos(z-z_i) + d(z)\sin(z-z_i)]^2\,\Delta^2_{\gamma_M,i}(k)\\
&- 2z_i^{-1}[a(z)\cos(z-z_i) + c(z)\sin(z-z_i)]\,[b(z)\cos(z-z_i) + d(z)\sin(z-z_i)]\,\Xi_i(k)\Big\}
\end{aligned} \tag{4.2.4b}$$

$$\begin{aligned}
\Xi(z,k) &= -z\Big\{[\alpha(z)\sin(z-z_i) + \lambda(z)\cos(z-z_i)]\,[a(z)\cos(z-z_i) + c(z)\sin(z-z_i)]\,\Delta^2_{\delta_M,i}(k)\\
&+ z_i^{-2}[\beta(z)\sin(z-z_i) + \gamma(z)\cos(z-z_i)]\,[b(z)\cos(z-z_i) + d(z)\sin(z-z_i)]\,\Delta^2_{\gamma_M,i}(k)\\
&- z_i^{-1}\{[\alpha(z)\sin(z-z_i) + \lambda(z)\cos(z-z_i)]\,[b(z)\cos(z-z_i) + d(z)\sin(z-z_i)]\\
&\quad + [\beta(z)\sin(z-z_i) + \gamma(z)\cos(z-z_i)]\,[a(z)\cos(z-z_i) + c(z)\sin(z-z_i)]\}\,\Xi_i(k)\Big\}
\end{aligned} \tag{4.2.4c}$$

where the following auxiliary functions are defined

$$\alpha(z) := \frac{1}{z_i} - \frac{1}{z} + \frac{1}{zz_i^2}, \qquad \beta(z) := 1 + \frac{1}{zz_i}, \qquad \lambda(z) := 1 + \frac{1}{zz_i} - \frac{1}{z_i^2}, \qquad \gamma(z) := \frac{1}{z} - \frac{1}{z_i} \tag{4.2.5a}$$

$$\begin{aligned}
a(z) &:= \frac{1}{z_i} - \frac{1}{z} + \frac{1}{zz_i^2} - \frac{1}{z^2 z_i}, \quad b(z) := 1 + \frac{1}{zz_i} - \frac{1}{z^2},\\
c(z) &:= -1 - \frac{1}{z^2} - \frac{1}{z_i^2} - \frac{1}{zz_i} - \frac{1}{z^2 z_i^2}, \quad d(z) := \frac{1}{z_i} - \frac{1}{z} - \frac{1}{z^2 z_i}
\end{aligned} \tag{4.2.5b}$$

with[2]

$$z = \frac{ak}{\sqrt{C}} = \frac{ak}{\sqrt{3\Omega_r}H_0 a_0^2} \tag{4.2.6}$$

and where $\Delta^2_{\delta_M,i}$, $\Delta^2_{\gamma_M,i}$ and Ξ_i are the initial values, at a_i (scale factor at end of inflation/beginning of radiation dominated era) of this power spectrum coefficients. Continuity is assured as $\lambda(z_i) = b(z_i) = 1$ and $\lambda(z_i) = a(z_i) = 0$.

Perturbations with $k > 1.876 \cdot 10^{22}$ Mpc^{-1} (in any case corresponding to extremely small scales and thus not statistically relevant) start the Radiation Dominated Era inside the horizon, while those with $k < 0.01047$ Mpc^{-1} will remain outside the horizon during the duration of this era. For those modes with k between the two, they will be progressively reentering the horizon and start their evolution.
As using eq. 2.3.13, we can approximate $\mathcal{H}(a) \approx H_0\sqrt{\Omega_r}\frac{a_0}{a}$, z can be expressed as

$$z = \frac{1}{\sqrt{3}a_0}\frac{k}{\mathcal{H}(a)} = \frac{1}{\sqrt{3}a_0}\frac{c\mathcal{H}^{-1}}{k^{-1}}, \tag{4.2.7}$$

being of the same order as the ratio between the Hubble horizon $c\mathcal{H}^{-1}$ and the perturbation wavelength. For superhorizon scales we will have $z \ll 1$, and for modes deep inside the horizon, $z \gg 1$.

[2] We remind the reader that this z variable is **not** a redshift, but rather an intermediate variable encapsulating both the scale factor and the wavenumber k. It can easily be related to the α term in the $\omega = -1$ case (section 4.4), as both appear while solving an analogue to the wave equation, with a real argument in the $\omega = \frac{1}{3}$ case, and an imaginary one in the $\omega = -1$ one.

4.2.1. Perturbations inside the horizon for $\omega = \frac{1}{3}$. From the expressions given in eqs. 4.2.3 for $\hat{\delta}_M$ it is easy to see that these perturbations, as well as the associated power coefficients oscillate around 0, with their amplitude changing according to the auxiliary functions defined above (expressions 4.2.5a and 4.2.5b). It is important to remember that we are dealing with the Fourier transform of these perturbations, rather than their "real" expressions, so these oscillations are of each of the k-modes, not of the particular perturbations at a certain $\vec{x}$ position. As the $\hat{\gamma}_M$ expression features an additional $z \propto a$ factor, its value (and those of $\Delta^2_{\gamma_M}$ and Ξ) progressively increases while oscillating. As we shall see in the following subsection, this additional term is only relevant for modes inside the horizon.
As it can be inferred from their expressions, the auxiliary functions feature an asymptotic behavior for $z \gg z_i$ (inside the horizon):

$$\begin{gathered}\lim_{z\to\infty} \alpha(z) = \lim_{z\to\infty} a(z) = \lim_{z\to\infty} |\gamma(z)| = \lim_{z\to\infty} d(z) = \frac{1}{z_i} \\ \lim_{z\to\infty} \lambda(z) = 1 - \frac{1}{z_i^2}, \qquad \lim_{z\to\infty} c(z) = -1 - \frac{1}{z_i^2}, \qquad \lim_{z\to\infty} \beta(z) = \lim_{z\to\infty} b(z) = 1\end{gathered} \tag{4.2.8}$$

This means that for $z \gg 1 \gg z_i$ (deep inside the horizon) these functions behave approximately as constants, so that in this case the only evolution expected for the power coefficients is the oscillating behavior from the sine and cosine functions (as well as the z term present in $\hat{\gamma}_M$). As the different k modes progressively reenter the horizon along this radiation dominated epoch, it is possible that some of the perturbations do not feature large enough values of z for this "limit" behavior to be reached.

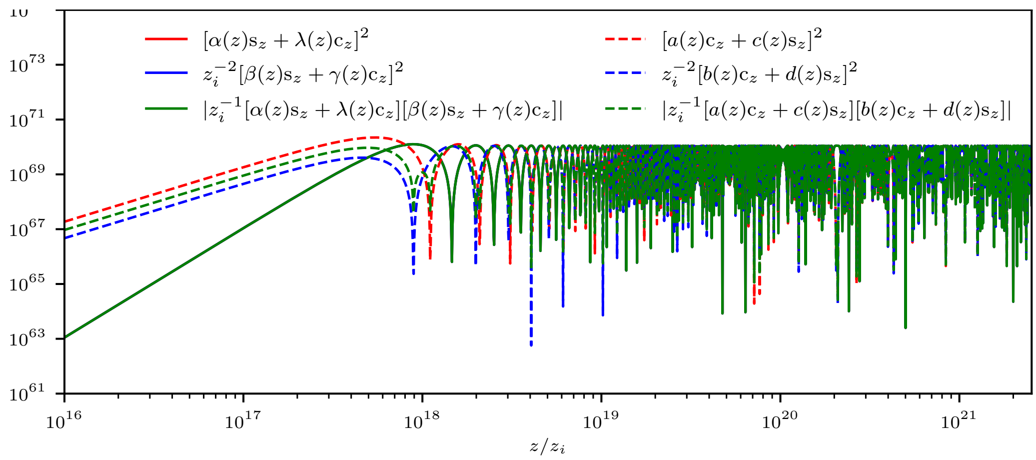

FIGURE 6. Plot of the functions accompanying each initial power coefficient $\{\Delta^2_{\delta_M,i}(k), \Delta^2_{\gamma_M,i}(k), \Xi_i(k)\}$ for $\Delta^2_{\delta_M}$ and $\Delta^2_{\gamma_M}$. The apparent difference in depth of the oscillations is a numerical issue due to taking logarithms for the representation, when the value approaches 0. Due to size problems, in the legend $\mathrm{c}_z = \cos(z - z_i)$ and $\mathrm{s}_z = \sin(z - z_i)$. Notice that z values are represented with respect to z_i (at the end of inflation).

Concerning now the oscillatory sine and cosine terms, the squared terms accompanying each initial power coefficient result in a sum of $\sin^2(z - z_i)$, $\cos^2(z - z_i)$ and $\sin(z - z_i)\cos(z - z_i) = \frac{1}{2}\sin[2(z - z_i)]$ terms, multiplied by a combination of the auxiliary functions defined above. Of these terms, the one featuring first the oscillating behavior is $\sin[2(z - z_i)]$. This way we can obtain the condition for the values of k for which the power coefficients feature oscillations. As $\sin(x)$ changes sign at $x = \pi$, we then have that the power coefficients will change their sign if

$$z_r - z_i \geq \frac{\pi}{2} \Leftrightarrow k \geq k_{osc} := \frac{\pi}{2(a_r - a_i)}\sqrt{3\Omega_r} H_0 a_0^2 c^{-1} \approx \frac{\pi}{2a_r}\sqrt{3\Omega_r} H_0 a_0^2 c^{-1} = 0.01983\ \mathrm{Mpc}^{-1} \tag{4.2.9}$$

so that if $a_r \gg a_i$, k_{osc} will be independent from the value of the scale factor at the end of inflation, which is greatly dependent on different parameters such as the number of e-folds or the approximately constant inflationary Hubble parameter H_*. This way, given $k \geq k_{osc}$, oscillating behavior will be present for $a \geq a_{osc}$, with

$$a_{osc} = a_i + \frac{\pi}{2k}\sqrt{3\Omega_r} H_0 a_0^2 c^{-1} < a_i + \frac{\pi}{2k_{osc}}\sqrt{3\Omega_r} H_0 a_0^2 c^{-1} = a_r \tag{4.2.10}$$

It is easy to see that the scale factor at which a k mode reenters the horizon is $a_{reenter} \approx \frac{\sqrt{\Omega_r} H_0 a_0}{ck}$, so it can be assumed that this oscillation begins shortly after reentering the horizon, with $a_{osc} \approx \frac{\sqrt{3}\pi}{2} a_{reenter} \approx 2.72 a_{reenter}$. Those with $k \in (0.01047, 0.01983)$ Mpc^{-1} do not suffer oscillations despite reentering the horizon along this epoch.

Regarding the physical interpretation of this behavior, as δ_M (equivalently $\Delta^2_{\delta_M}$) oscillates around a constant value and in the radiation dominated region $\bar{\rho}(a)$ evolves as $\mathcal{O}(a^{-4})$, there is a oscillating decay in the amplitude of the Bardeen potential $\Phi_H \propto a^2 \bar{\rho}(a) \delta_M$, in which the gravitational growth of the density inhomogeneities is not strong enough to counter the fast dilution of the average density, meaning that behaves as a damped oscillator while its amplitude decreases as a^{-2}.
Those perturbation modes with $k < 0.01047$ Mpc$^-1$ (large ones) will not experience any oscillation, as will remain outside the horizon during this era.

4.2.2. Perturbations outside the horizon for $\omega = \frac{1}{3}$. We will now see how the superhorizon limit $k \ll c^{-1}\mathcal{H}$ matches the behavior described in section 3.3. As we commented above, in this limit $z \ll 1$. As $\frac{a_i}{\sqrt{3\omega} H_0 a_0^2} = 3.065 \cdot 10^{-20}$ Mpc, we can assume that for our most of the Radiation Dominated Era (even if outside the horizon) $z \gg z_i \sim 0$. This way the sine and cosine functions behave as

$$\sin(z - z_i) \approx z, \qquad \cos(z - z_i) \approx 1 + \frac{z^2}{2} \tag{4.2.11}$$

We are taking into account the second order term of the cosine, as its arguments take values between 0 and 1 on superhorizon scales, and the $\cos x \approx 1$ approximation is only valid for arguments close to 0. Applying these approximations to eqs. 4.2.3, and using only the leading growing terms, we find that density perturbations behave on superhorizon scales as

$$\hat{\delta}_M \approx \underbrace{\frac{1}{zz_i}}_{\mathcal{O}(a^{-1})} (\hat{\delta}_{M,i} + \hat{\gamma}_{M,i}) + \underbrace{\frac{3z}{2z_i}}_{\mathcal{O}(a)} (\hat{\delta}_{M,i} + \hat{\gamma}_{M,i}) - \underbrace{\frac{z^2}{2z_i^2}}_{\mathcal{O}(a^2)} \left[(1 - z_i)\hat{\delta}_{M,i} + \hat{\gamma}_{M,i}\right] \sim -\frac{z^2}{2z_i^2}(\hat{\delta}_{M,i} + \hat{\gamma}_{M,i}) \tag{4.2.12}$$

Similarly, for the power perturbations,

$$\begin{aligned} \hat{\gamma}_M \quad &\approx \quad - \underbrace{\frac{1}{zz_i}}_{\mathcal{O}(a^{-1})} (\hat{\delta}_{M,i} + \hat{\gamma}_{M,i}) - \underbrace{2}_{\mathcal{O}(1)} \hat{\delta}_{M,i} + \underbrace{\frac{z}{2z_i}}_{\mathcal{O}(a)} (-\hat{\delta}_{M,i} + \hat{\gamma}_{M,i}) - \underbrace{\frac{z^2}{2z_i^2}}_{\mathcal{O}(a^2)} \left[(1 - 3z_i)\hat{\delta}_{M,i} + 3\hat{\gamma}_{M,i}\right] \\ &+ \underbrace{\frac{z^3}{2z_i}}_{\mathcal{O}(a^3)} (\hat{\delta}_{M,i} + \hat{\gamma}_{M,i}) \sim -\frac{z^2}{2z_i^2}(\hat{\delta}_{M,i} + 3\hat{\gamma}_{M,i}) \end{aligned} \tag{4.2.13}$$

As the relation between $\frac{z^3}{z_i}$ and $\frac{z^2}{z_i^2}$ is the same as between z and z_i^{-1}, and as $z_i \ll 1$, $z_i^{-1} \gg 1 \gg z$, the $\mathcal{O}(a^2)$ term is much more dominant than the $\mathcal{O}(a^3)$. This way it is clear that (while under important corrections) for superhorizon scales $\delta_M \propto a^{3\omega+1} = a^2$. If we additionally suppose that $\hat{\gamma}_{M,i} \sim \hat{\delta}_{M,i}$ (the power and density initial perturbations have similar magnitude), then it is clear that $\gamma_M \approx 2\delta_M$.

4.2.3. General comments for $\omega = \frac{1}{3}$. Now that the overall evolution trends for the evolution equations have been studied, we can concentrate on how the initial conditions $\{\Delta^2_{\delta_M,i}(k), \Delta^2_{\gamma_M,i}(k), \Xi_i(k)\}$ affect the evolution of the different power coefficients. First of all, it is easy to see that for $z \gg z_{i_*}$, the auxiliary functions can be paired by their asymptotic behavior, namely $a(z) \sim \alpha(z)$, $b(z) \sim \beta(z)$, $c(z) \sim |\lambda(z)|$ and $d(z) \sim |\gamma(z)|$. By inspecting eqs. 4.2.4a and 4.2.4b, we find that, without taking into account the initial z^2 factor for $\Delta^2_{\gamma_M}$, their expressions are analogue, changing only the sine and cosine, what, as it will be seen, does not have much effect in the limit $z \gg z_i$. As it can be seen in Figure 6, while for lower z the two families of functions are several orders of magnitude apart, once the oscillating part starts, they quickly converge to the same behavior.

Using this, we can reach the following conclusions on the behavior of the power coefficients:

- It is immediate to see that, for a given $z > z_i$, that is $a > a_i$, and for every k mode,

$$\Delta^2_{\delta_M}(a, k) \propto \left[\Delta^2_{\delta_M,i}(k) + \Delta^2_{\gamma_M,i}(k) - 2\Xi_i(k)\right], \qquad \text{with } a \text{ fixed} \tag{4.2.14}$$

this means, that, while the evolution of the power coefficient associated with the matter/energy density varies with a (or equivalently z) as depicted by the solid line in Figure 6, the importance of the initial coefficients does not change.

- Regarding $\Delta^2_{\gamma_M}$, as its expression is preceded by a z^2 term, it will not experience an oscillating behavior with constant amplitude, as do the functions displayed in Figure 6, but rather will keep growing as it oscillates. As it is evident by Figure 7 and the limit discussion in the previous subsection, the behavior outside the horizon is the same (save for a 4 factor) as in $\Delta^2_{\delta_M}$. As we are using as initial conditions $\Delta^2_{\gamma_M,i} = 4\Delta^2_{\delta_M,i}$ and $\Xi_i = 2\Delta^2_{\delta_M,i}$, considering adiabatic initial curvature perturbations, the initial contributions are fixed and of similar importance, but could be different if other initial perturbations were considered.
- Finally, for $\Xi(z,k)$, which, as it has been defined it has an intermediate behavior between $\Delta^2_{\delta_M}(a,k)$ and $\Delta^2_{\gamma_M}(a,k)$, so it will present the effects of both coefficients.

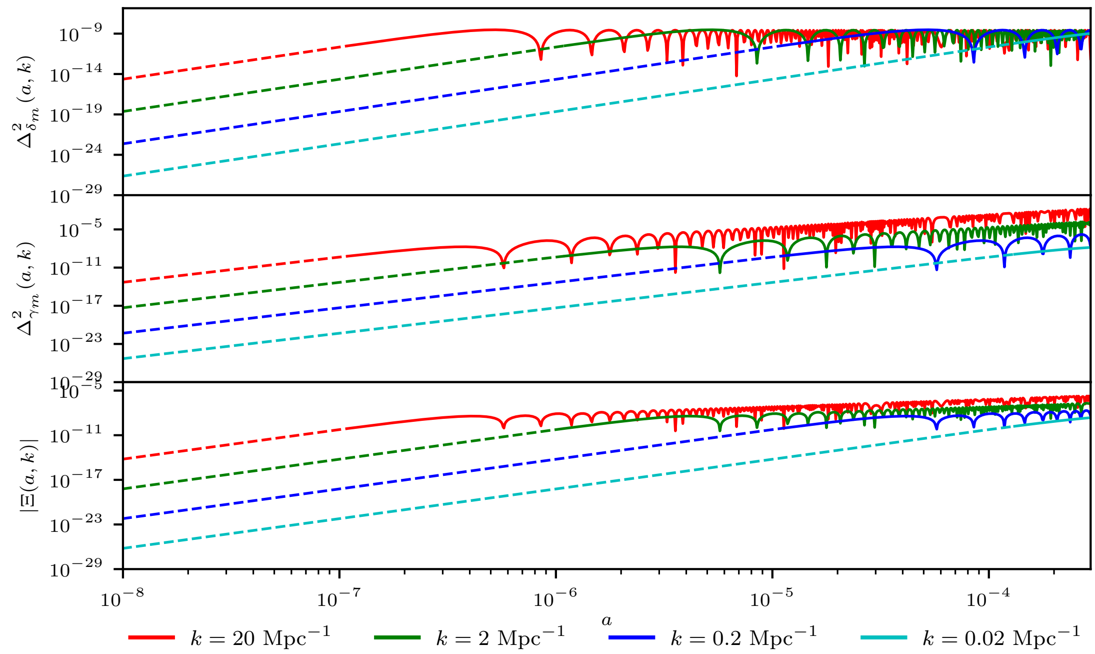

FIGURE 7. Power coefficients $\Delta^2_{\delta_M}(a,k)$, $\Delta^2_{\gamma_M}(a,k)$ and $\Xi(a,k)$ as a function of a, for several Fourier modes $|k|$, as indicated by the legend under the plots, during the radiation dominated era of the Universe. The initial conditions for the curvature perturbations are $\Delta^2_{\mathcal{R}}(k) = 2\cdot 10^{-9}$ for all the modes. The evolution of the perturbations while outside the horizon is given in dashed lines.

Once we have indicated the behavior of the contributions from each initial condition, we can represent the full evolution of $\Delta^2_{\delta_M}(a,k)$, $\Delta^2_{\gamma_M}(a,k)$ and $\Xi(a,k)$ as a grows with the expansion of the Universe, during the radiation dominated era. To do this, the evolution of the power coefficients for different initial conditions and Fourier modes have been represented in Figure 7, reaching the following conclusions:

- Regarding the power spectrum of the density perturbations, $\Delta^2_{\delta_M}$, they all feature a similar behavior along their growth, featuring an oscillating behavior shortly after reentering the horizon, which is dependent only on k (larger k modes will reenter earlier). For the same initial conditions for $\Delta_{\mathcal{R}^2}$, the amplitude of these oscillations is k independent, as the auxiliary functions inside the horizon are dominated by $z_i^{-2} \propto k^{-2}$ (limit values of $\lambda(z)$ and $c(z)$), and the relation from $\mathcal{R}$ to $\hat{\delta}_M$ and $\hat{\gamma}_M$ through Φ_H involves a k^2 term (eq. 3.2.5).
- In the case of the power perturbation terms, $\Delta^2_{\gamma_M}$ also start oscillating shortly after they reenter the horizon, being in counterphase with $\Delta^2_{\delta_M}$, as the sin and cos terms are interchanged as γ_M is obtained from the derivative of δ_M. As one would expect by comparing with the expression of the density perturbations, $\Delta^2_{\gamma_M}$ grows with a due to the z^2 term multiplying their expressions. This is

interesting, as larger k-modes will have higher values for $\Delta^2_{\gamma_M}$ at a_r, serving as initial values for the Matter Dominated Era.

- Regarding the cross-correlation coefficients, Ξ starts oscillating shortly after crossing the horizon, with an increasing amplitude when the perturbations start to evolve. Unlike $\Delta^2_{\gamma_M}$, the cross-correlation features a z factor (instead of z^2), so this amplitude increase is slower, as evidenced in Figure 7.

It is important to notice that, regardless of the initial conditions, the $\Delta^2_{\gamma_M}(a,k)$ coefficients reach much higher values than the $\Delta^2_{\delta_M}(a,k)$ ones, an important fact to consider when calculating the initial conditions for the Matter Dominated Era.

Our derivations reproduce the evolution behavior for $\hat{\delta}_M$ obtained from the standard treatment (see section 5.2.1 from [**Baumann, 2016**]), up to the dominant term, this is $\hat{\delta}_M \sim \lambda(z)\cos(z)\delta_{M,i}$, but our approach is novel in that it incorporates lower order terms (both for the $\hat{\delta}_{M,i}$ and $\hat{\gamma}_{M,i}$ initial conditions), as well as the "phase mismatch" $z - z_i$. We also provide a lower limit $k_{osc} = 0.01274$ Mpc^{-1} for the perturbation modes which experiment oscillations along this era, as well as the relation between the horizon reentry and the start of the oscillating scale factors, $a_{osc} \approx 2.72\, a_{reenter}$.

4.3. Epoch II: Matter Dominated Era ($\omega = 0$)

During the Matter Dominated Era (from $a_r = 2.952 \cdot 10^{-4}$ to $a_m = 0.767$) the Universe can be considered to be composed by non-interacting, non-relativistic dust particles, with a $\omega = 0$ equation of state. In this case, equation 4.1.9 takes the following form:

$$a^2 \frac{\partial^2 \varepsilon}{\partial a^2} + \frac{3}{2}\left(a\frac{\partial \varepsilon}{\partial a} - \varepsilon\right) = 0 \tag{4.3.1}$$

This is a much simpler equation than in other cases, as $\omega = 0$ removes the $\nabla^2\varepsilon$ term from equation 4.1.9, meaning that the solutions of eq. 4.3.1 depend only on a and not on their position, so it evolution equation takes the form of an homogeneous ODE. As explained on Appendix C, the δ_M and γ_M coefficients can be easily obtained over the real space (that is, depending on their position $\vec{x}$ rather than their Fourier mode k), having the following polynomial expression:

$$\delta_M(a,\vec{x}) = \delta_{M,r}(\vec{x})\left[\frac{3}{5}\left(\frac{a}{a_r}\right) + \frac{2}{5}\left(\frac{a}{a_r}\right)^{-3/2}\right] + \gamma_{M,r}(\vec{x})\left[\frac{2}{5}\left(\frac{a}{a_r}\right) - \frac{2}{5}\left(\frac{a}{a_r}\right)^{-3/2}\right] \tag{4.3.2a}$$

$$\gamma_M(a,\vec{x}) = \delta_{M,r}(\vec{x})\left[\frac{3}{5}\left(\frac{a}{a_r}\right) - \frac{3}{5}\left(\frac{a}{a_r}\right)^{-3/2}\right] + \gamma_{M,r}(\vec{x})\left[\frac{2}{5}\left(\frac{a}{a_r}\right) + \frac{3}{5}\left(\frac{a}{a_r}\right)^{-3/2}\right] \tag{4.3.2b}$$

where the scale factor a_r corresponds to the scale factor at the end of the radiation dominated era. It is easy to see that the two initial perturbations $\delta_{M,r}(\vec{x})$ and $\gamma_{M,r}(\vec{x})$ contribute to the growth of both perturbations, whose expressions converge rapidly when $a \gg a_r$ as

$$\delta_M(a,\vec{x}) \approx \gamma_M(a,\vec{x}) \approx \frac{1}{5}\left[3\delta_{M,r}(\vec{x}) + 2\gamma_{M,r}(\vec{x})\right]\frac{a}{a_r} \tag{4.3.3}$$

meaning that in this case both perturbations tend to the same value, growing linearly with the same intensity.
Now the perturbations described in equations 4.3.2 have the following associated coefficients:

$$\begin{aligned}\Delta^2_{\delta_M}(a,k) = \langle\hat{\delta}_M(a,k)^2\rangle = \frac{1}{25}\Bigg\{&\Delta^2_{\delta_M,r}(k)\left[9\left(\frac{a}{a_r}\right)^2 + 12\left(\frac{a}{a_r}\right)^{-1/2} + 4\left(\frac{a}{a_r}\right)^{-3}\right]\\ +\Delta^2_{\gamma_M,r*}(k)\left[4\left(\frac{a}{a_r}\right)^2 - 8\left(\frac{a}{a_r}\right)^{-1/2} + 4\left(\frac{a}{a_r}\right)^{-3}\right] &+ \Xi_r(k)\left[12\left(\frac{a}{a_r}\right)^2 - 4\left(\frac{a}{a_r}\right)^{-1/2} - 8\left(\frac{a}{a_r}\right)^{-3}\right]\Bigg\}\end{aligned} \tag{4.3.4a}$$

$$\begin{aligned}\Delta^2_{\gamma_M}(a,k) = \langle\hat{\gamma}_M(a,k)^2\rangle = \frac{1}{25}\Bigg\{&\Delta^2_{\delta_M,r}(k)\left[9\left(\frac{a}{a_r}\right)^2 - 18\left(\frac{a}{a_r}\right)^{-1/2} + 9\left(\frac{a}{a_r}\right)^{-3}\right]\\ +\Delta^2_{\gamma_M,r}(k)\left[4\left(\frac{a}{a_r}\right)^2 + 12\left(\frac{a}{a_r}\right)^{-1/2} + 9\left(\frac{a}{a_r}\right)^{-3}\right] &+ \Xi_r(k)\left[12\left(\frac{a}{a_r}\right)^2 + 6\left(\frac{a}{a_r}\right)^{-1/2} - 18\left(\frac{a}{a_r}\right)^{-3}\right]\Bigg\}\end{aligned} \tag{4.3.4b}$$

$$\Xi(a,k) = \langle \hat{\delta}_M(a,k)\hat{\gamma}_M(a,k)\rangle = \frac{1}{25}\left\{\Delta^2_{\delta_M,r}(k)\left[9\left(\frac{a}{a_{r*}}\right)^2 - 3\left(\frac{a}{a_r}\right)^{-1/2} - 6\left(\frac{a}{a_r}\right)^{-3}\right]\right.$$
$$\left. +\Delta^2_{\gamma_M,r}(k)\left[4\left(\frac{a}{a_r}\right)^2 + 2\left(\frac{a}{a_r}\right)^{-1/2} - 6\left(\frac{a}{a_r}\right)^{-3}\right] + \Xi_r(k)\left[12\left(\frac{a}{a_r}\right)^2 + \left(\frac{a}{a_r}\right)^{-1/2} + 12\left(\frac{a}{a_r}\right)^{-3}\right]\right\} \tag{4.3.4c}$$

where $\Delta^2_{\delta_M,r}$, $\Delta^2_{\gamma_M,r}$ and Ξ_r are the initial values of this power spectrum coefficients at $a = a_r$.

Perturbation modes with $k > 0.01407$ Mpc^{-1} are already inside the horizon when the Matter Dominated Era commences. As during this epoch the conformal Hubble parameter reaches its minimum at $a_{min} = 0.6091 < a_m$, the Hubble horizon reaches its maximum, $c\mathcal{H}^{-1}_{max} = 5010.02$ Mpc (from eq. 2.3.13), associated to the minimum wavenumber of the perturbations modes which at some point will be able to reenter the horizon, $k_{min} = 0.000196$ Mpc^{-1}. Those modes with $k < k_{min}$ will not reenter the horizon at any point during the expansion of the Universe. By the end of the Matter Dominated Epoch at $a = a_m = 0.767$, those modes with $k < 0.000205$ Mpc^{-1} the perturbation modes will have re-exited the horizon and thus will not follow the above equations anymore.

4.3.1. Perturbations inside the horizon for $\omega = 0$. Given the expression for δ_M, the power coefficients feature a polynomial behavior, with the $\mathcal{O}(a^2)$ term dominating over $\mathcal{O}(a^{-1/2})$ and $\mathcal{O}(a^{-3})$, so that for $a \gg a_r$, the three coefficients, analogously to what happened in expression 4.3.3, converge to

$$\Delta^2_{\delta_M}(a,k) \approx \Delta^2_{\gamma_M}(a,k) \approx \Xi(a,k) \approx \frac{1}{25}\left[9\Delta^2_{\delta_M,r}(k) + 4\Delta^2_{\gamma_M,r}(k) + 12\Xi_r(k)\right]\left(\frac{a}{a_r}\right)^2 \tag{4.3.5}$$

For those modes with $a_m \gg a_r$ (which would be the case for those which are inside the horizon having reentered it during this era or those modes which entered in the previous one, for $a \gg a_r$), the final value of the perturbations at the end of the matter dominated era (which in turn will determine the initial values for the next perturbation evolution) is given by the $(\frac{a}{a_r})^2$ term.

4.3.2. Perturbations outside the horizon for $\omega = 0$. As the evolution of the perturbations is k independent, horizon crossing condition $k = c^{-1}\mathcal{H}$ cannot be used to study a certain limit behavior. However, it is easy to see that our equations satisfy the superhorizon qualitative behavior derived in section 3.3 for $\omega \neq -1$, in which case Φ_H freezes outside the horizon and thus $\delta_M \propto (a^2\bar{\rho}(a))^{-1} \propto a^{3\omega+1}$ and $\gamma_M = (1+3\omega)\delta_M$. In our case, it is clear that this is in fact what happens, with a $\mathcal{O}(a^{3/2})$ correction:

$$\delta_M = \left(\frac{a}{a_r}\right)\left(\frac{3}{5}\delta_{M,r} + \frac{2}{5}\gamma_{M,r}\right) + \frac{2}{5}\left(\frac{a}{a_r}\right)^{-3/2}(\delta_{M,r} - \gamma_{M,r}) \tag{4.3.6a}$$

$$\gamma_M = \left(\frac{a}{a_r}\right)\left(\frac{3}{5}\delta_{M,r} + \frac{2}{5}\gamma_{M,r}\right) + \frac{3}{5}\left(\frac{a}{a_r}\right)^{-3/2}(\gamma_{M,r} - \delta_{M,r}) \approx \delta_M \tag{4.3.6b}$$

As it is obvious, even if $\delta_{M,r}$ and $\gamma_{M,r}$ have not similar values, the $\mathcal{O}(a^{-3/2})$ term (which can be interpreted as being associated with the $a^{-\frac{3\omega+5}{2}} = a^{-5/2}$ decaying term in Φ_H as given by expression 3.3.14) decays sharply, and shortly after entering the Matter Dominated Era only the growing mode $\mathcal{O}(a)$ is relevant, corresponding with what was expected from our discussion for the superhorizon limit.
Regarding now the power coefficients $\Delta^2_{\delta_M}$, $\Delta^2_{\gamma_M}$ and Ξ, a small comment must be made for $a \gtrapprox a_r$, where the $\mathcal{O}(a^{-1/2})$ and $\mathcal{O}(a^{-3})$ terms might be important, we have to study the accompanying functions of the initial coefficients for each case. Denoting $x := \frac{a}{a_r}$, we have that the nine polynomial functions from eqs. 4.3.4 feature monotonous growth for x considerably greater than 1. While this is the case for the whole dominion of those accompanying $\Delta^2_{\delta_M}$, it is not the case for $\Delta^2_{\gamma_M}$ (where the function associated with $\Delta^2_{\gamma_M,r}$ has a minimum at $x = 1.383$, where it takes a value of 0.6207 compared to the initial value of 1) and Ξ (where that accompanying Ξ_r has a minimum at $x = 1.088$, taking a value of 0.9792 compared to the initial value of 1). While this behavior might be interesting, it has no important effects, as it occurs shortly after this epoch starts and the "minimum" is of small relative magnitude.

4.3.3. General comments for $\omega = 0$. In Figure 8 the evolution of the three perturbation coefficients are plotted. As the wavenumber k does not appear in the coefficients expression, every Fourier mode evolves the same way (unlike what happened during the radiation dominated era), given the same initial perturbations.

As we had discussed, there is an overall monotonous growth of the power coefficients (once the perturbations are inside the horizon) which have the same behavior for every k mode once evolution starts. It is also important to notice that, as can be seen in the $k = 0.02$ Mpc^{-1}, and thus have different initial conditions than the rest, the power coefficients converge quite rapidly to the dominant value, doing so with a steep increase.

Again, while the evolution is independent from k, as perturbation modes with higher k reenter the horizon earlier (or are even already inside it), they have more time to grow, which means that these perturbations will end up being several orders of magnitude bigger than other perturbations which reenter the horizon later. Regarding the reentry of the perturbation modes which start this era outside the horizon, as in both cases the growth is a polynomial one (be it of degree 4 outside the horizon and 2 inside), the power coefficients transition smoothly, as evidenced in figure 8.

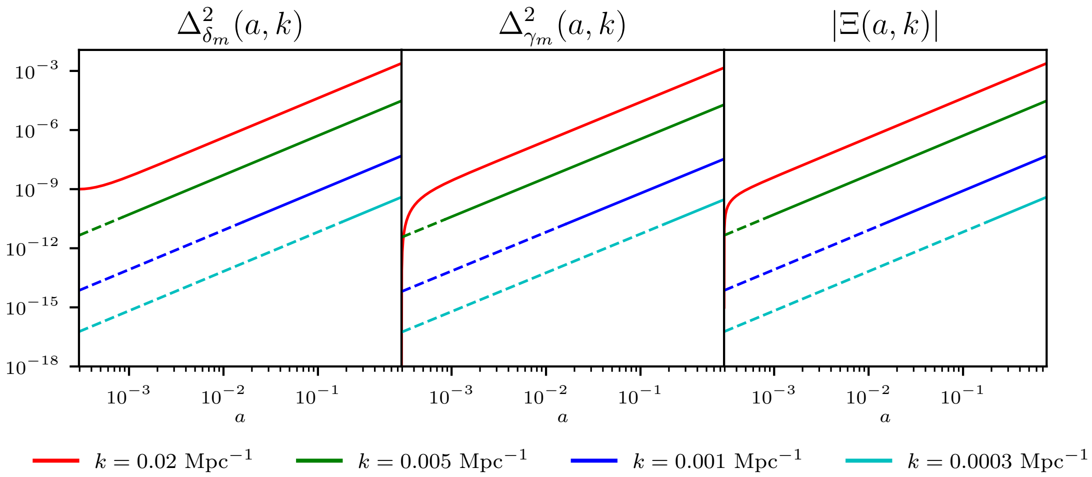

FIGURE 8. Power coefficients $\Delta^2_{\delta_M}(a,k)$, $\Delta^2_{\gamma_M}(a,k)$ and $\Xi(a,k)$ as a function of a, for a single Fourier mode $|k|$, during the radiation dominated era of the Universe. The initial conditions are $\Delta^2_{\mathcal{R}}(k) = 2 \cdot 10^{-9}$ for the modes starting outside the horizon, and $\Delta^2_{\delta_M,r} = 10^{-9}$, $\Delta^2_{\gamma_M,r} = 10^{-21}$ and $\Xi_r = 10^{-15}$ for that inside ($k = 0.02$ Mpc-1). The evolution of the perturbations while outside the horizon is given in dashed lines.

In this case we reproduce the behavior of the δ_M perturbations as can be found in [**Baumann, 2016**] and [**Piattella, 2018**] (though in this case the $\mathcal{O}(a^{-3/2})$ is not found). While the expresions in both texts are k-independent, it is not explicitly stated, though it is an immediate implication of a k-free evolution equation. One important feature is that, as only the two solution modes are given in [**Baumann, 2016**], the contributions of the initial δ_M and γ_M is not given, unlike in our approach.

4.4. Solution for $\omega = -1$

During two periods of its history, the Universe can be considered to be governed by a $\omega = -1$ equation of state. These are respectively the first and last "relevant" eras of cosmic evolution: Inflation Era (ranging from $a_p = 1.02 \cdot 10^{-53}$ to $a_i = 1.165 \cdot 10^{-27}$) and our current Dark Energy Dominated Era (ranging from $a_m = 0.767$ to $a_0 = 1$). As $\omega = -1$, eq. 4.1.9 takes the following form in the real space, resulting in an elliptic PDE ($-a^2 \frac{3}{C} a^{-2} = -3C^{-1} < 0$):

$$a^2 \frac{\partial^2 \varepsilon}{\partial a^2} + \frac{3}{C} a^{-2} \nabla^2 \varepsilon = 0, \tag{4.4.1}$$

and in the Fourier space:

$$a^2 \frac{\partial^2 \hat{\varepsilon}}{\partial a^2} - \frac{3}{C} a^{-2} k^2 \hat{\varepsilon} = 0, \tag{4.4.2}$$

regarding the constant C appearing in the above equations, we have $C = \Lambda = 3H^2\Omega_\Lambda$, with $H = H_*$ and $\Omega_\Lambda = 1$ (completely exponential growth) for the inflationary, and $H = H_0$ and $\Omega_\Lambda = 0.6889 \pm 0.0056$ for the Λ-dominated case.

These equations describe the evolution of the gauge-invariant comoving density perturbations ε, which are related with the gauge-invariant fractional density perturbations by $\varepsilon = \bar{\rho}(a)a^3\delta_M$. During $\omega = -1$ epochs, however, this distinction is subtler. If we consider the $\omega = -1$ behavior of the Universe along this epoch to be caused by a Λ cosmological constant (such as in the current state of the Universe), which is the same in every point of the Universe, then it makes no sense consider perturbations in the associated density $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$. As the Universe is also composed by matter and radiation, the perturbations will be on this "underlying" density $\bar{\rho}_{eff}$, be it dominated by radiation or matter, which evolves according to its equation of state ω_{eff}. On the other hand, as we explained in the introductory chapter, inflation can be explained by the existence of an (scalar) field ϕ, with an associated stress-energy tensor. We could consider its density ρ_ϕ being subject to possible perturbations $\delta\rho_\phi$, so the $\omega_{eff} = -1$ scenario could still be possible.

Additional considerations must be made regarding the evolution of the perturbations outside the horizon, as we must take into account the distinction between the background, dynamic-driving $\omega = -1$ and $\bar{\rho}$ dominated by a cosmological constant/inflation field, and the underlying fluid susceptible to perturbations, with equation of state ω_{eff} and density $\bar{\rho}_{eff}$.

As described by equation 3.3.17, in a $\omega = -1$ evolving Universe, the Bardeen potential Φ_H evolves as a^{-1} outside the horizon, rather than remaining frozen as is the case when $\omega \neq -1$. Furthermore depending on the value of ω_{eff} the dominant density of the fluid susceptible to perturbations can change, so the relation between δ_M and Φ_H by means of the Poisson's Equation (eq. 3.2.5) is affected. It is easy to show then that outside the horizon the fractional density perturbations evolve as

$$\left.\begin{aligned}\Phi_H(a) = \Phi_H(a_{out})\frac{a_{out}}{a} &= \frac{4\pi G}{k^2}\frac{a_{out}^3}{a}\bar{\rho}_{eff}(a_{out})\delta_M(a_{out}) \\ \bar{\rho}_{eff}(a) &= \bar{\rho}_{eff}(a_{out})\left(\frac{a}{a_{out}}\right)^{-3(1+\omega_{eff})}\end{aligned}\right\} \Rightarrow \delta_M(a) = \delta_M(a_{out})\left(\frac{a}{a_{out}}\right)^{3\omega_{eff}} \tag{4.4.3}$$

Regarding the relation between δ_M and γ_M outside the horizon, we must note that eq. 4.1.17 is no longer valid, as Φ_H is not constant, as stated by eq. 3.3.17. However, this problem is immediately solved using eq. 4.4.3:

$$\delta_M(a) = \frac{\partial\delta_M}{\partial\ln a} = a\frac{\partial\delta_M}{\partial a} = 3\omega_{eff}\delta_M(a_{out})\left(\frac{a}{a_{out}}\right)^{3\omega_{eff}} = 3\omega_{eff}\delta_M(a), \tag{4.4.4}$$

so that $\gamma_M(a) = 3\omega_{eff}\delta(a)$ and

$$\Delta^2_{\gamma_M} = 9\omega_{eff}^2\Delta^2_{\delta_M^2}, \qquad \Delta^2_{\gamma_M} = 3\omega_{eff}\Delta^2_{\delta_M^2} \tag{4.4.5}$$

We will show along this section the solutions for $\omega_{eff} \in \{-1, 0, \frac{1}{3}\}$, obtaining general solutions, with $H \in \{H_*, H_0\}$, $a_* \in \{a_p, a_m\}$ and $\Omega_\Lambda \in \{1, 0.6889 \pm 0.0056\}$, depending on whether we are considering the inflationary or Dark Energy dominated case. Furthermore, we will define the following quantity, which appears in an analogous way to the z in the $\omega = \frac{1}{3}$ case (see Appedix C):

$$\alpha := \frac{k}{\sqrt{\Omega_\Lambda}}H_0^{-1}\left(\frac{1}{a} - \frac{1}{a_*}\right) \tag{4.4.6}$$

As during inflation H_* is considered constant, $\mathcal{H} = H_* a$ and during the Λ-dominated Era $\mathcal{H}(a) \approx H_0\sqrt{\Omega_\Lambda}\frac{a}{a_0}$ (see eq. 2.3.13), we can interpret $\alpha = \frac{kc}{\mathcal{H}(a)} - \frac{kc}{\mathcal{H}(a_*)}$ as the difference between the quotients of the perturbation wavelength k^{-1} and the Hubble comoving horizon $c\mathcal{H}(a)^{-1}$. For perturbations approaching the horizon and entering superhorizon scales, $kc \lesssim \mathcal{H}(a)$, and α approaches its limit value, being bounded:

$$\lim_{a\to\infty}\alpha(a) = -\frac{k}{H_*}\frac{1}{a_p} \Longrightarrow |\alpha| \in \left[0, \frac{k}{H_*}\left(\frac{1}{a_p} - \frac{1}{a_i}\right)\right] \subset \left[0, \frac{k}{H_*}\frac{1}{a_p}\right] \tag{4.4.7}$$

We will denote by $\delta_{M,*}(k)$, $\gamma_{M,*}(k)$, $\Delta^2_{\delta_M,*}$, $\Delta^2_{\gamma_M,*}$ and Ξ_* the initial values of the perturbations and power coefficients at a_*.

4.4.1. Effective Equation of State $\omega_{eff} = -1$. The perturbation solutions associated to eq. 4.4.2 for an underlying fluid of equation of state $\omega_{eff} = -1$ are the following

$$\hat{\delta}_M(a,k) = \left(\frac{a_*}{a}\right)^2\left[\left(\cosh|\alpha| + \frac{2Ha_*}{k}\sinh|\alpha|\right)\hat{\delta}_{M,*}(k) + \frac{Ha_*}{k}\sinh|\alpha|\gamma_{M,*}(k)\right] \tag{4.4.8a}$$

$$\begin{aligned}\hat{\gamma}_M(a,k) = & -\frac{k}{Ha}\left(\frac{a_*}{a}\right)^2\left\{\left[\frac{2H}{k}(a-a_*)\cosh|\alpha| - \left(1 - \frac{4H^2}{k^2}a_*a\right)\sinh|\alpha|\right]\hat{\delta}_{M,*}(k)\right. \\ & \left. -\frac{Ha_*}{k}\left[\cosh|\alpha| + \frac{2H_*a}{k}\sinh|\alpha|\right]\hat{\gamma}_{M,*}(k)\right\}\end{aligned} \tag{4.4.8b}$$

with the following associated power coefficients:

$$\begin{aligned}\Delta^2_{\delta_M}(a,k) &= \left(\frac{a_*}{a}\right)^4\left[\left(\cosh|\alpha| + \frac{2Ha_*}{k}\sinh|\alpha|\right)^2\Delta^2_{\delta_M,*}(k) + \frac{H^2a_*^2}{k^2}\sinh^2|\alpha|\Delta^2_{\gamma_M,*}(k)\right.\\ &\left.+\frac{Ha_*}{k}\left(\sinh|2\alpha| + \frac{4Ha_*}{k}\sinh^2|\alpha|\right)\Xi_*(k)\right]\end{aligned} \tag{4.4.9a}$$

$$\begin{aligned}\Delta^2_{\gamma_M}(a,k) &= \left(\frac{a_*}{a}\right)^4\frac{k^2}{H^2a^2}\left\{\left[\frac{2H}{k}(a-a_*)\cosh|\alpha| - \left(1-\frac{4H^2}{k^2}a_*a\right)\sinh|\alpha|\right]^2\Delta^2_{\delta_M,*}(k)\right.\\ &+\frac{H^2a_*^2}{k^2}\left[\cosh|\alpha| + \frac{2Ha_*}{k}\sinh|\alpha|\right]^2\Delta^2_{\gamma_M,*}(k)\\ &\left.-2\frac{Ha_*}{k}\left[\frac{2H}{k}(a-a_*)\cosh|\alpha| - \left(1-\frac{4H^2}{k^2}a_*a\right)\sinh|\alpha|\right]\left[\cosh|\alpha| + \frac{2H}{k}a\sinh|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{4.4.9b}$$

$$\begin{aligned}\Xi(a,k) &= \left(\frac{a_*}{a}\right)^4\frac{k}{Ha}\left\{\left(\cosh|\alpha| + \frac{2Ha_*}{k}\sinh|\alpha|\right)\left[\frac{2H}{k}(a-a_*)\cosh|\alpha| - \left(1-\frac{4H^2}{k^2}a_pa\right)\sinh|\alpha|\right]\Delta^2_{\delta_M,*}(k)\right.\\ &\left.+\frac{H^2a_*^2}{k^2}\left[\frac{1}{2}\sinh|2\alpha| + \frac{2H}{k}a\sinh^2|\alpha|\right]\Delta^2_{\gamma_M,*}(k) - \frac{Ha_*}{k}\left[\cosh^2|\alpha| + \sinh^2|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{4.4.9c}$$

4.4.2. Effective Equation of State $\omega_{eff} = 0$. We now consider that, while the Universe undergoes an evolution governed by a $\omega = -1$ equation of state, the dominant fluid susceptible to perturbations has an equation of state $\omega_{eff} = 0$, this is, dust-like matter.
The evolution of these perturbations while inside the horizon are the following:

$$\hat{\delta}_M(a,k) = \frac{a}{a_*}\left[\left(\cosh|\alpha| - \frac{Ha_*}{k}\sinh|\alpha|\right)\hat{\delta}_{M,*}(k) + \frac{Ha_*}{k}\sinh|\alpha|\hat{\gamma}_{M,*}(k)\right] \tag{4.4.10a}$$

$$\hat{\gamma}_M(a,k) = -\frac{k}{Ha}\frac{a}{a_*}\left\{\left[\frac{H}{k}(a_*-a)\cosh|\alpha| + \left(\frac{H^2a_*a}{k^2}-1\right)\sinh|\alpha|\right]\hat{\delta}_{M,*}(k) - \frac{Ha_*}{k}\left[\cosh|\alpha| + \frac{Ha}{k}\sinh|\alpha|\right]\hat{\gamma}_{M,*}(k)\right\}, \tag{4.4.10b}$$

with the following associated power coefficients:

$$\begin{aligned}\Delta^2_{\delta_M}(a,k) &= \left(\frac{a}{a_*}\right)^2\left[\left(\cosh|\alpha| - \frac{Ha_*}{k}\sinh|\alpha|\right)^2\Delta^2_{\delta_M,*}(k) + \left(\frac{Ha_*}{k}\right)^2\sinh^2|\alpha|\Delta^2_{\gamma_M,*}(k)\right.\\ &\left.-2\frac{Ha_*}{k}\left(\frac{Ha_*}{k}\sinh^2|\alpha| - \frac{1}{2}\sinh|2\alpha|\right)\Xi_*(k)\right]\end{aligned} \tag{4.4.11a}$$

$$\begin{aligned}\Delta^2_{\gamma_M}(z,k) &= \left(\frac{k}{Ha}\right)^2\left(\frac{a}{a_*}\right)^2\left\{\left[\frac{H}{k}(a_*-a)\cosh|\alpha| + \left(\frac{H^2a_*a}{k^2}-1\right)\sinh|\alpha|\right]^2\Delta^2_{\delta_M,*}(k)\right.\\ &+\left(\frac{Ha_*}{k}\right)^2\left[\cosh|\alpha| + \frac{Ha}{k}\sinh|\alpha|\right]^2\Delta^2_{\gamma_M,*}(k)\\ &\left.-2\frac{Ha_*}{k}\left[\frac{H}{k}(a_*-a)\cosh|\alpha| + \left(\frac{H^2a_*a}{k^2}-1\right)\sinh|\alpha|\right]\left[\cosh|\alpha| + \frac{Ha}{k}\sinh|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{4.4.11b}$$

$$\begin{aligned}\Xi(a,k) &= -\frac{k}{Ha}\left(\frac{a}{a_*}\right)^2\left\{\left(\cosh|\alpha| - \frac{Ha_*}{k}\sinh|\alpha|\right)\left[\frac{H}{k}(a_*-a)\cosh|\alpha| + \left(\frac{H^2a_*a}{k^2}-1\right)\sinh|\alpha|\right]\Delta^2_{\delta_M,*}(k)\right.\\ &+\left(\frac{Ha_*}{k}\right)^2\left[\frac{1}{2}\sinh|2\alpha| + \frac{Ha}{k}\sinh^2|\alpha|\right]\Delta^2_{\gamma_M,*}(k)\\ &\left.-\frac{Ha_*}{k}\left[\cosh^2|\alpha| + \frac{H}{k}(a-a_*)\sinh|2\alpha| + \left(1-\frac{2Ha_*}{k^2}\right)\sinh^2|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{4.4.11c}$$

4.4.3. Effective Equation of State $\omega_{eff} = \frac{1}{3}$. We will finally consider the case in which the Universe is governed by a $\omega = -1$ equation of state, with the fluid susceptible to experience perturbations dominated by radiation ($\omega_{eff} = \frac{1}{3}$ equation of state). The perturbations evolution while inside the horizon is given by

$$\hat{\delta}_M(a,k) = \left(\frac{a}{a_*}\right)^2\left[\left(\cosh|\alpha| - \frac{2Ha_*}{k}\sinh|\alpha|\right)\hat{\delta}_{M,*}(k) + \frac{Ha_*}{k}\sinh|\alpha|\hat{\gamma}_{M,*}(k)\right] \tag{4.4.12a}$$

$$\hat{\gamma}_M(a,k) = \frac{k}{Ha}\left(\frac{a}{a_*}\right)^2\left\{\left[\frac{2H}{k}(a_*-a)\cosh|\alpha| - \left(1-\frac{4Ha_*a}{k^2}\right)\sinh|\alpha|\right]\hat{\delta}_{M,*}(k) - \frac{Ha_*}{k}\left[\cosh|\alpha| + \frac{2Ha}{k}\sinh|\alpha|\right]\hat{\gamma}_{M,*}(k)\right\} \tag{4.4.12b}$$

The associated power coefficients are the following:

$$\begin{aligned}\Delta^2_{\delta_M}(a,k) \;=\; & \left(\frac{a}{a_*}\right)^4\left[\left(\cosh|\alpha| - \frac{2Ha_*}{k}\sinh|\alpha|\right)^2\Delta^2_{\delta_M,*}(k) + \left(\frac{Ha_*}{k}\right)^2\sinh^2|\alpha|\Delta^2_{\gamma_M,*}(k)\right.\\ & \left. -\frac{Ha_*}{k}\left(\sinh|2\alpha| - \frac{4Ha_*}{k}\sinh^2|\alpha|\right)\Xi_*(k)\right]\end{aligned} \tag{4.4.13a}$$

$$\begin{aligned}\Delta^2_{\gamma_M}(a,k) \;=\; & \left(\frac{k}{Ha}\right)^2\left(\frac{a}{a_*}\right)^4\left\{\left[\frac{2H}{k}(a_*-a)\cosh|\alpha| - \left(1-\frac{4H^2a_*a}{k^2}\right)\sinh|\alpha|\right]^2\Delta^2_{\delta_M,*}(k)\right.\\ & +\left(\frac{Ha_*}{k}\right)^2\left[\cosh|\alpha| + \frac{2Ha}{k}\sinh|\alpha|\right]^2\Delta^2_{\gamma_M,*}(k)\\ & \left.-2\frac{Ha_*}{k}\left[\frac{2H}{k}(a_*-a)\cosh|\alpha| - \left(1-\frac{4H^2a_*a}{k^2}\right)\sinh|\alpha|\right]\left[\cosh|\alpha| + \frac{2Ha}{k}\sinh|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{4.4.13b}$$

$$\begin{aligned}\Xi(z,k) \;=\; & \frac{k}{Ha}\left(\frac{a}{a_*}\right)^4\left\{\left(\frac{2Ha_*}{k}\sinh|\alpha| - \cosh|\alpha|\right)\left[\frac{2H}{k}(a_*-a)\cosh|\alpha| - \left(1-\frac{4H^2a_*a}{k^2}\right)\sinh|\alpha|\right]\Delta^2_{\delta_M,*}(k)\right.\\ & +\left(\frac{Ha_*}{k}\right)^2\left[\frac{1}{2}\sinh|2\alpha| + \frac{2Ha}{k}\sinh^2|\alpha|\right]\Delta^2_{\gamma_M,*}(k)\\ & \left.+\frac{Ha_*}{k}\left[\cosh^2|\alpha| + \tfrac{2Ha}{k}(a_*-a)\sinh|2\alpha| + \left(1-\frac{8H^2a_*a}{k^2}\right)\sinh^2|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{4.4.13c}$$

4.4.4. Some considerations and problems about our equation and solutions for $\omega = -1$. Once we have our solutions or the $\omega = -1$ evolution equation (eq. 4.4.1), some comments need to be made. For simplicity's sake, we will discuss the evolution equations for δ_M and γ_M, rather than the associated power coefficients, as their expressions are shorter. Nonetheless, the same conclusions can be immediately extrapolated to $\Delta^2_{\delta_M}$, $\Delta^2_{\gamma_M}$ and Ξ due to the way these are defined.

We will begin by noticing that expressions 4.4.8, 4.4.10a and 4.4.12 have very similar expressions, namely a $(\frac{a}{a_*})^{3\omega_{eff}+1}$ term multiplying a series of hyperbolic terms, differing only in some sign or 2 factor between them. This is appears as a result of the equations coming from a common solution for ε, which are then expressed in terms of δ_M and γ_M by means of the relation $\varepsilon = a^3\bar{\rho}_{eff}(a)\delta_M$. The different signs or factors between them appear due to first derivatives with respect to a being used when expressing δ_M and γ_M in terms of their initial conditions. As

$$\frac{Ha_*}{c} = \begin{cases} 2.280\cdot 10^{-4} & \text{during Inflation}\\ 1.748\cdot 10^{-4} & \text{during }\Lambda\text{-dominated Era}\end{cases} \tag{4.4.14}$$

for modes with $k > 0.001$ Mpc^{-1} the perturbations for the three cases approximately behave as

$$\hat{\delta}_M \sim \left(\frac{a}{a_*}\right)^{3\omega_{eff}+1}\cosh|\alpha|\hat{\delta}_{M,*}, \qquad \hat{\gamma}_M \sim (3\omega_{eff}+1)\left(\frac{a}{a_*}\right)^{3\omega_{eff}+1}\cosh|\alpha|\hat{\delta}_{M,*}, \tag{4.4.15}$$

were only the growing terms have been considered for γ_M. This way the general behavior of the perturbations consists in the following two phases:

- At first there is a short but intense quasi-exponential growth given by the hyperbolic functions, and dominated by $\cosh|\alpha|$. We have that $|\alpha|$ is a function of a which takes positive values, ranging from 0 at $a = a_*$ to its limit value of $\frac{k}{\sqrt{\Omega_\Lambda}Ha_*}$. As $\cosh x$ and $\sinh x$ can be accurately approximated by $\frac{1}{2}e^x$ for values with $x \gg 1$ the growth of the perturbations is dominated by a $e^{|\alpha|}$ term during this first phase. It could be argued that this exponential growth of the perturbations in counter-intuitive, as one would expect that during the quasi-exponential expansion of the Universe during the $\omega = -1$ dominated eras, density perturbations would be rapidly dissipated due to this rapid expansion.
- As α approaches its limit value, it ceases growing with a, which means that the hyperbolic functions behave as a constant. As we explained, the limit $\lim_{a\to\infty}\alpha(a)$ corresponds to the super-horizon case ($k \ll c^{-1}\mathcal{H}(a)$), so it makes sense that this exponential growth stops once the scale of the perturbations is outside the Hubble sphere. The perturbations evolution is then dominated by a $(\frac{a}{a_*})^{3\omega_{eff}+1}$ evolution, corresponding this to a decay or a growth in their magnitude depending on the value of ω_{eff}.

While these solutions might seem reasonable at first sight, there are some problems with them which must acknowledged:

- First of all, the superhorizon behavior is not in line with our derivations for perturbations evolving during a $\omega = -1$ dominated era, during which the Bardeen potential evolved as $\Phi_H(a) \propto a^{-1}$, and as a result the density perturbations did so as $\delta_M \propto a^{3\omega_{eff}}$. Instead we are left with a $\delta_M \propto a^{3\omega_{eff}+1}$ behavior, which is quite incompatible with our previous derivations.
- Furthermore, while the exponential growth of the perturbations might not seem problematic at first, we can consider the following. In the Λ-dominated case, were the primary perturbed fluid is the underlying matter ($\omega_{eff} = 0$), as in the ΛCDM model Λ is an universal constant and thus not subject to perturbations, we would be dealing with an important increase in the fractional density perturbations, ranging from 30 for $k = 0.002$ Mpc^{-1} to $4 \cdot 10^{1394}$ for 2 Mpc^{-1}, all of this during the relatively short time encompassing from $a_m = 0.767$ to $a_0 = 1$.

While this behavior could be interpreted in the sense that large scale structure is being formed, with overdensities being particularly important for smaller scales (larger k), which could account for the formation of large scale structure such as matter around galaxies, galaxy clusters, etc., we must take into account that the derivation of the evolution equations present in chapter 3 were of linear nature. For large overdensities, linear perturbations could be no longer valid, as higher order terms would be relevant. While some progress has been made regarding the treatment of higher order perturbations in General Relativity (see [**Nakamura, 2019**] for an idea of the state of the art on second order Cosmological Perturbation Theory, and the mathematical complication it entails), in practice semi-classic approaches are preferred, which put into practice numerical modes to solve N-body problems.
In any case, our main problem with our solutions seems to be that they do not agree with those obtained in [**Baumann, 2016**], page 107. These, which are derived from the perturbed Einstein Equations $\delta G^i_j = \delta T^i_j$ (eqs. 3.3.5c) under $\bar{P}\pi_L \approx 0$ perturbations approximation (for late times it is sensible to assume that pressure perturbations can be neglected, as the non-gravitational interaction between galaxies could be neglected), imply that the matter density perturbation modes remain constant in later (Λ-dominated) times. This way, linear evolution stops, with galaxy clusters ceasing to form, and all the subsequent evolution given by higher order perturbation terms. This behavior seems more plausible than that from our solutions, as large scale structure does not seem to be growing exponentially at the current moment. The same conclusion can be reached from eq. 3.2.9a with $\bar{P} = -\bar{\rho}$ and $\pi_L = 0$, which results in

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar{\rho}_{eff} a^3 \delta_M) = 0 \tag{4.4.16}$$

As in the matter case $\bar{\rho}_{eff} \propto a^{-3}$, we have $\dot{\delta}_M = 0$, meaning that δ_M remains constant for all scales during this era. This is a nice argumentation which uses important simplifications, but the complete evolution entails important points which must be discussed.

The two problems listed above seem to indicate that, as our solutions were directly obtained from Bardeen's evolution equation without any additional assumption, equation 3.2.26 might not be appropriate to describe the evolution of perturbations on a $\omega = -1$ dominated Universe. While its expression does not feature any term which could be ill-defined for $\omega = -1$, such as $\frac{1}{\omega+1}$, while deriving Bardeen's equation we made use of the momentum equation 3.2.9b, which features $\frac{\omega}{\omega+1}$ factors for π_L and π_T, the first of which we are not setting to zero, as in adiabatic fluctuations $\pi_L = c_s^2 \omega^{-1} \delta$, meaning that under our simplifying assumptions ($c_s^2 = \omega$ and $\delta = \delta_M$ in time-orthogonal gauge), $\pi_L = \delta_M$. We would need a version of eq. 3.2.26, derived under the $\omega = -1$ assumption, that enables us to obtain the complete evolution of perturbations, specially considering that we are currently living in a Λ-dominated Era in which the background dynamics of the Universe are governed by a $\omega = -1$ equation of state.
There exist additional considerations that must be taken when trying to solve this problem. As derived in Appendix B (eq. B.3.27), the $\omega = -1$ analogue to the momentum equation is

$$\pi_L = \left(1 - \frac{3K}{k^2}\right) \pi_T, \tag{4.4.17}$$

with both equations come from the same expression (eq. B.3.25). As when working under the adiabatic assumption, $\pi_L = \frac{c_s^2}{\omega}\delta_M$, if we considered the anisotropic stress to be zero, $\pi_T = 0$, then the δ_M perturbations would be null. This would imply that, in order to work with scalar perturbations under $\omega = -1$, we would need to introduce additional perturbations such as $\eta \neq 0$ or $\pi_T \neq 0$. Even if Bardeen's Equation (eq. 3.2.26) were appropriate for a $\omega = -1$ expansion, this would give way to an equation more complex than eq. 4.1.9.

Finally, regarding the possible perturbations appearing due to the scalar field dynamics during inflation, as the stress-energy tensor associated to the scalar field ϕ is $T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta} - V(\phi)\right)$, it is quite straightforward to show that if ϕ is only a function of time,

$$\omega(\phi) = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}, \tag{4.4.18}$$

meaning that the equation of state can vary smoothly with ϕ depending on the potential $V(\phi)$ and the "kinetic energy" $\frac{1}{2}\dot{\phi}^2$. Considering different fluctuations of ϕ, anisotropic and non-adiabatic perturbations could appear in $T_{\mu\nu}$. In any case, studying this would be a considerable enterprise and thus we will leave it for a future work.

4.5. Complete evolution of scalar perturbations

While in the previous sections of this chapter we studied and described the different solutions of the evolution equation (eq. 4.1.9) for different equations of state along the evolution of the Universe, in this section we will obtain the complete evolution of initial scalar perturbations, discussing the main features present.
As we described at the beginning of this chapter, we have worked under the approximation that the Universe can be described as being governed by the equation of state ω of the dominating component at each epoch. This is supported by Figure 3, by which it is evident that, while this approximation is weaker around the transition from one era to the next, it nonetheless allows us to obtain "simple" analytic expressions. Starting from the initial conditions obtained from the measured spectrum of curvature perturbations $\Delta^2_{\mathcal{R}}(k)$ given by 2.4.12, we will obtain the evolution of the power coefficients for the density and power perturbations, as well as the cross-correlation, along the different eras, using the final perturbations of each era as the initial conditions for the evolution during the following. We will assume that the transition between one another is fast and not very drastic, as can be inferred from Figure 3, so that the actual final and initial conditions are not very different from those from our approach.

From our results, we will study both the complete evolution of the power coefficients, as well as that concerning only the "growing" (dominant) modes, and assess the importance of the corrections coming from "decaying" (subdominant) modes.

4.5.1. Complete evolution coming from $\mathcal{R}$ initial perturbations. We will firstly consider the already discussed initial perturbations coming from the quasi-scale invariant curvature perturbations $\mathcal{R}$, whose measured power spectrum is given by

$$\Delta^2_{\mathcal{R}}(k) = 2.445 \cdot 10^{-9} \left(\frac{k}{k_*}\right)^{-0.0374}, \qquad \text{with} \quad k_* = 0.05 \text{ Mpc}^{-1} \tag{4.5.1}$$

Though this parametrization is given in [**Planck Col.-Infl., 2018**] for modes with $k \in [0.008, 0.1]$ Mpc^{-1}, we will extrapolate the expression for a larger interval of k. These perturbations are then translated to Φ_H by eq. 3.3.15, and to density and power perturbations by 3.2.5 and 4.1.19. As we discussed earlier, this means that the initial perturbations are not independent but, as we have seen, it is an acceptable approximation when working with adiabatic perturbations ($\eta = 0$), for which $\mathcal{R}$ perturbations freeze for superhorizon scales. The evolution for several k modes is plotted in Figure 9.

As it can be seen, those modes with higher k reenter the horizon earlier (as this happens when $k = c^{-1}\mathcal{H}$, and $\mathcal{H}$ decreases with time for intermediate eras). As it can be seen in Figure 9 (and was mentioned in section 4.2), those modes with $k < 0.01047$ Mpc^{-1} are outside the horizon during the complete duration of the Radiation Era, and thus will not experience the characteristic oscillation at the end of this epoch. The same way, these "higher" perturbation modes seem to feature larger initial (both density and power) perturbations, as $\delta_M \propto k^2\Phi_H$ for a certain a. This is countered by the fact that higher k modes correspond to smaller perturbation wavelengths k^{-1}, and are thus less statistically significant.

We now regard the end of the Radiation Era, which features the most interesting behavior of the different evolution regimes, it is important to comment how. Those modes which are inside the horizon during this time oscillate, with their density perturbations doing so around a fixed amplitude, while the perturbations outside the horizon continue growing. This can be reverted (see $k = 2$ Mpc^{-1} mode in Figure 9) when transitioning to the Matter Dominated Era, where the three coefficients converge to the same behavior, where the power perturbation contribution dominates over the others, as the $\Delta^2_{\gamma_M}$ coefficient keeps growing while oscillating at the end of the radiation dominated era.

On the contrary, this power perturbation coefficient can be associated to a negative power perturbation (understanding this as $\gamma_M = \frac{\partial \delta_M}{\partial \ln a} < 0$), so that this contribution is not as important in this case, as it must "reverse" its sign when entering the Matter Dominated Epoch (while the power coefficients are always positive, the perturbations they are associated with can take negative values, meaning an underdensity or a decrease in this density perturbation). See the $k = 0.2$ Mpc^{-1} in the following Figure as an example of this.

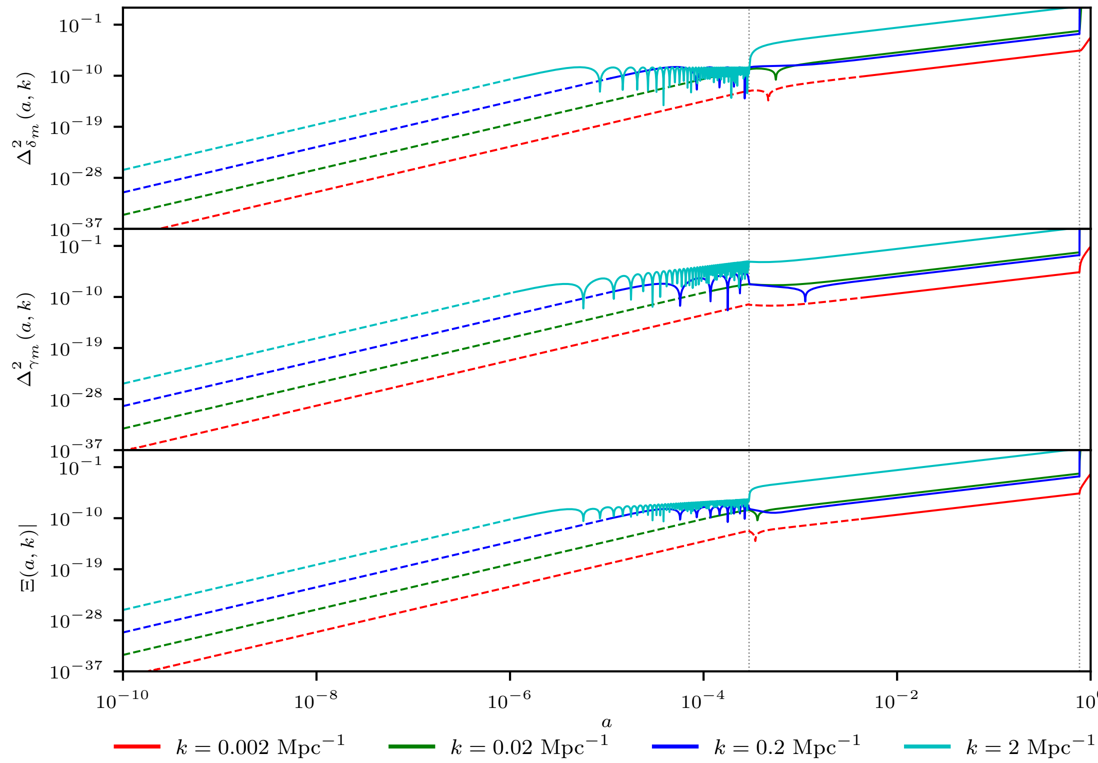

FIGURE 9. Complete evolution for several k modes of the power coefficients of density and power pertubations, as well as cross-correlation coefficients. The evolution of the perturbations while outside the horizon is given in dashed lines. The limits between each of theepochs is represented by vertical, dotted lines.

As we explained in the previous section, our approach seems not to be valid during the Λ-dominated era as, as apart from the possibility that linear perturbations are no longer valid, there are discrepancies between our exponential solutions and the constant ones given by [**Baumann, 2016**]. Because of this, we will refrain from trying to obtain any definite conclusion from perturbations during this era. It will require future work to study the cause of this important differences.

We could ask ourselves if, along our results, the linear perturbation approximation is valid, or on the contrary, at some point overdensities become so important that higher orders must be used. While deriving the evolution equation (eq. 3.2.26), we implicitly assumed that our metric perturbations coefficients $\{A, B, H_L, H_T\}$ were all small enough, so that only the linear terms were relevant. As $\Phi_H \sim H_L$, and $\Phi_H \propto \frac{a^2}{k^2}\bar{\rho}_{eff}(a)\delta_M$ it would seem that those modes with lower k would in turn have larger values for the associated Bardeen potential. However, as $\delta_{M,i} \propto k^2(a^2\bar{\rho}_{eff})^{-1}\Phi_H(a_i)$, with $\Phi_H(a_i)$ scale invariant, the magnitude of Φ_H with k will be determined by the effect k has in the evolution of the density perturbations. Though evolution during Matter Dominated Era is k-independent, it is biased towards modes with high k, as seen in Figure 10, due to the contribution of the power perturbations γ_M, which keep growing during the Radiation Dominated Era. This way, we can conclude that the Bardeen potential power spectrum modes will be larger for larger k, for which in turn linear approach will cease being valid earlier. This corresponds with smaller scales, i.e. galaxies, as opposed to galaxy clusters (featuring as smaller overall overdensity).

While Figure 9 offers a nice visualization of the evolution of specific perturbation modes along the expansion of the Universe, it may be also convenient to plot a portion of the perturbations power spectrum for certain scale factors a, as in Figure 10, where power coefficients corresponding to perturbation modes with $k \in [5 \cdot 10^{-4}, 5]$ Mpc^{-1} have been plotted for $a \in \{10^{-5}, 10^{4}, 10^{-3}, 0.1\}$.

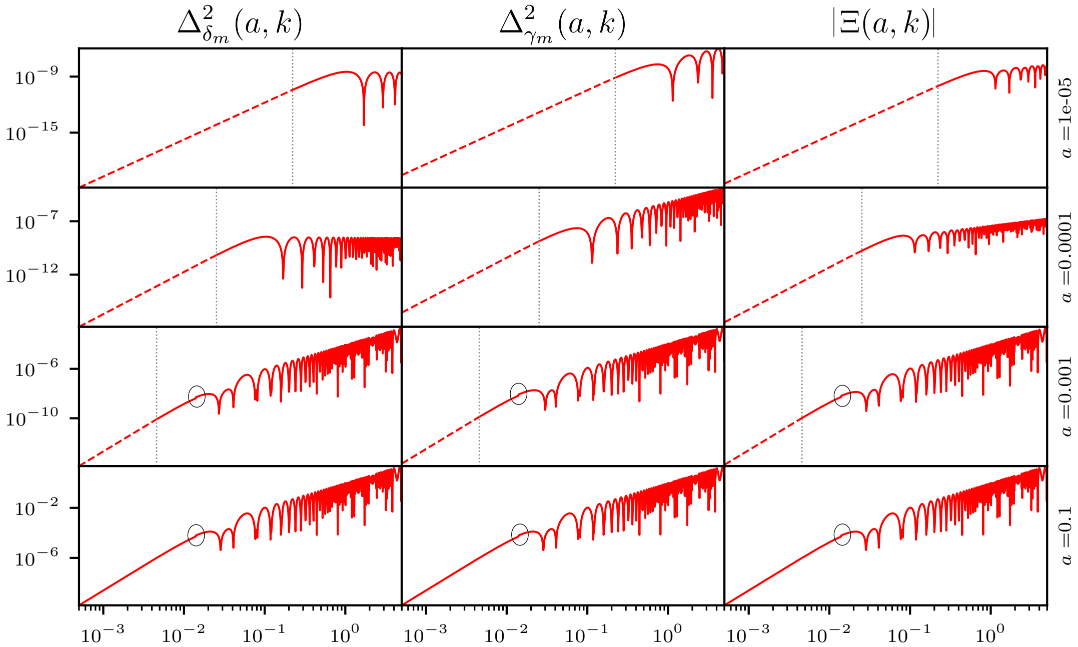

FIGURE 10. Power spectrum coefficients for the density and power perturbations, as well as the cross-correlation coefficients, for $k \in [5 \cdot 10^{-4}, 5]$ Mpc^{-1}, for selected scale factors a (indicated at the right side of each row). Those k modes which are outside the horizon for each case are plotted in a dashed line, with the reentry threshold marked by a vertical line. The discontinuity due to the 10% decrease in Φ_H when transitioning from the Radiation to Matter Dominated Eras is highlighted with a circle.

The first two rows from Figure 10, with $a = 10^{-5}, 10^{-4}$, correspond to two certain moments during the Radiation Dominated Epoch. As it can be seen, higher k modes progressively reenter the horizon, and their density perturbation power coefficients $\Delta^2_{\delta_m}$ start oscillating around the fixed amplitude for which they reentered the horizon. Considering an arbitrary power spectrum for the primordial curvature perturbations $\Delta^2_{\mathcal{R}}(k)$, it is easy to see that, when reentering the horizon at $a_{in}(k)$ (specific for each mode), we have

$$\Delta^2_{\delta_M}(k, a_{in}(k)) \propto \left[\frac{\mathcal{H}\left(a_{in}(k)\right)^2}{a_{in}(k)^2\bar{\rho}(a_{in}(k))}\right]\Delta^2_{\mathcal{R}}(k) = \left[1 + \frac{\Omega_K\left(\frac{a}{a_0}\right)^{-2}}{\Omega_m\left(\frac{a}{a_0}\right)^{-3} + \Omega_r\left(\frac{a}{a_0}\right)^{-2} + \Omega_\Lambda}\right]\Delta^2_{\mathcal{R}}(k) \approx \Delta^2_{\mathcal{R}}(k), \tag{4.5.2}$$

meaning that the magnitude of the power spectrum coefficients associated with density perturbations, $\Delta^2_{\delta_M}$, has the same relation with k as the initial curvature perturbations spectrum, $\Delta^2_{\mathcal{R}}(k)$. As shortly after reentering the horizon the magnitude of the density perturbations stops growing and starts oscillating, the amplitude of these oscillations in the density perturbation power coefficients is directly proportional to the amplitude of $\Delta^2_{\mathcal{R}}(k)$. As we have used a near scale-invariant initial curvature perturbations ($n_s = 0.9626$), the amplitudes of the oscillating coefficients during the Radiation Dominated Epoch (first two rows of Figure 10) are approximately constant.

It is also easy to see that, as time passes and more modes cross the horizon, the oscillation progresses as $z \propto ak$ (the argument in the sine and cosine functions describing the evolution of the perturbations, see eqs. 4.2.3 and 4.2.4) is proportional to a and k. Despite what might seem from Figure 10, as well as Figure 4.2.4, the period of this oscillations does not decrease exponentially with k (an "optical effect" caused by the logarithm scale in both axes), but rather decreases linearly with k^{-1}, so that perturbations with smaller wavelength oscillate faster than those with larger ones. This can be interpreted by the fact that, as the spatial amplitude of a perturbation associated to a mode k is of the k^{-1} order, the acoustic waves (which travel at sound velocity c_s) take more time to travel through the perturbed region. This results in a longer oscillation period, and a smaller frequency.

Regarding the power perturbation coefficients, their amplitude grows with k^2 due to the $z^2 \propto k^2$ term in these coefficients, which has the interpretation that higher modes oscillate more quickly, which in turn increases the "variation" of density perturbations. The same reasoning applies to the behavior of the cross-correlation coefficients Ξ, with the amplitude of its oscillations growing with k.

For the third and fourth rows the perturbations have already entered the Matter Dominated Era, so new modes reentering the horizon do not experience an oscillatory behavior, but rather continue their $\mathcal{O}(a^2)$ growth, which is scale independent. This way the plots in the last two rows differ only in a vertical shift along caused by this a^2 growth, as in both cases the scale factor is much greater than $a_r = 2.954 \cdot 10^4$, and the evolution is now proportional to a^2.

Regarding the evolution of the density perturbations modes which were inside the horizon during the Radiation Dominated Era, and thus experienced an oscillatory behavior during that epoch, we see in Figure 10 that the growth for the modes is bigger for those with greater k. This is due to the contribution of the power perturbations (here represented by the $\Delta^2_{\gamma_M}$ and Ξ coefficients), which keep growing during the Radiation Dominated Era, and are larger for bigger k values. Because of this contribution, modes with larger k have bigger amplitudes during the Matter Dominated Era.

One last comment about the transition from Radiation to Matter dominated is the apparent discontinuity in the power spectrum for $k = 0.01407\ \mathrm{Mpc}^{-1}$ (barely appretiable in the last two rows of Figure 10). This is caused by the transition in the relation between the curvature perturbations $\mathcal{R}$ and Bardeen potential Φ_H from $\Phi_H = \frac{2}{3}\mathcal{R}$ (for $\omega = \frac{1}{3}$) to $\Phi_H = \frac{3}{5}\mathcal{R}$ ($\omega = 0$), a decrease of a 10%, and must be forced "externally", as it is a relation between $\mathcal{R}$ and Φ_H, so it is not taking into account by the Bardeen's Equation (eq. 3.2.26), which only describes the evolution of the perturbations. In reality, the transition is not immediate, so this change happens smoothly, avoiding this discontinuity our approximation implies. This "smooth transition" is described in section 9.9.1 from [**Piattella, 2018**], though the discussion restricts to the transition of the Φ_H deduced from Friedmann Equations, rather than our evolution equation. In any case, taking into account this would probably make our expressions much more complex, so it will not be put into use in this work.

4.5.2. Proportions between different perturbations. Baryons. One remark that must be taken into account is that when obtaining the evolution and growth of the perturbations δ_M, we are working with *fractional* perturbations (in the time-orthogonal gauge $\delta_M = \delta$, so that $\rho = \bar{\rho} + \delta\rho = \bar{\rho}(1+\delta_M)$) and that, while $\delta_M(a)$ is growing in magnitude, $\bar{\rho}(a)$ is rapidly decaying. It must be also taken into account that, while we are considering that the Universe is dominated at each time by a single equation of state, both matter and radiation are present at each time, with their densities evolving differently. It is not difficult to see how the relative importance of each contribution changes with the evolution of the Universe. We will begin by assuming *adiabatic-like*[3] *perturbations*, i.e. for matter and radiation.

$$\frac{\delta\rho_r}{\bar{\rho}_r} = \frac{\delta\rho_m}{\bar{\rho}_m} \Rightarrow \begin{cases} \delta\rho_r &= \delta\rho_m \dfrac{\bar{\rho}_r}{\bar{\rho}_m} &= \delta\rho_m \dfrac{\Omega_r}{\Omega_m}\left(\dfrac{a}{a_0}\right)^{-1} \\ \delta\rho_m &= \delta\rho_r \dfrac{\bar{\rho}_m}{\bar{\rho}_r} &= \delta\rho_r \dfrac{\Omega_m}{\Omega_r}\left(\dfrac{a}{a_0}\right) \end{cases} \tag{4.5.3}$$

Considering now that for times after radiation only matter and radiation experience perturbations, $\delta_M = \bar{\rho}^{-1}\delta\rho = \bar{\rho}^{-1}(\delta\rho_m + \delta\rho_r)$, it is easy to see that

$$\delta_{M,r} := \frac{\delta\rho_r}{\bar{\rho}} = \delta_M\left[1 + \frac{\Omega_m}{\Omega_r}\left(\frac{a}{a_0}\right)\right]^{-1}, \qquad \delta_{M,m} := \frac{\delta\rho_m}{\bar{\rho}} = \delta_M\left[1 + \frac{\Omega_r}{\Omega_m}\left(\frac{a}{a_0}\right)^{-1}\right]^{-1}, \tag{4.5.4}$$

with $\delta_M = \delta_{M,r} + \delta_{M,m}$. As it can be deduced from Figure 11, there is an important difference in the contribution of the matter and radiation fractional densities to the total δ_M. As one would expect, the relative importance of each contribution is larger in the epochs their fluids dominate, with the two crossing at $a_r = 2.952 \cdot 10^{-4}$, at the end of the Radiation Dominated Era, when $\bar{\rho}_r(a_r) = \bar{\rho}_m(a_r)$.

[3]While the term adiabatic is reserved for initial fluctuations, in the absence of entropy perturbations η the subsequent perturbations preserve their defining property.

It is also very interesting to see that during the Matter Dominated Era the radiation fractional perturbations $\delta_{R,r}$ remain constant, as well as an additional mode rapidly decaying as $a^{-5/2}$. One important consequence of this fact is that, as Recombination occurs at $z_{rec} = 1100$ ($a_{rec} = 9.08 \cdot 10^{-4}$), during the Matter Dominated Era, so the different acoustic oscillations of the different modes for the radiation perturbations must come from those occurring during the Radiation Dominated Era (the decaying $\mathcal{O}(a^{-5/2})$ will have a small effect, but it can be neglected for this discussion). This is also present in the solutions obtained by [**Baumann, 2016**] (see section 5.2.1).

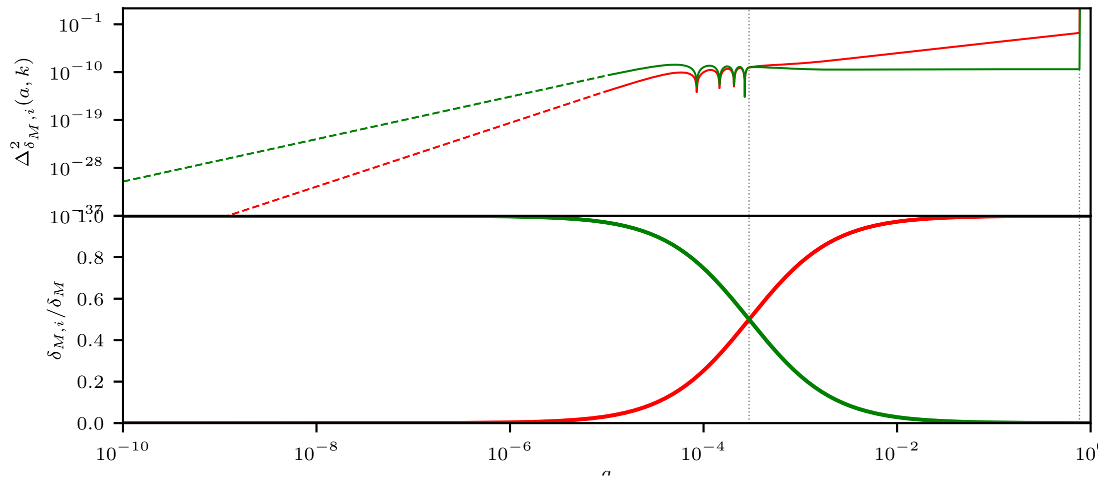

FIGURE 11. Power spectrum coefficients associated to the radiation and matter density perturbations ($\Delta^2_{\delta_{M,r}}$ and $\Delta^2_{\delta_{M,m}}$, respectively) for the $k = 0.2$ Mpc^{-1} mode. In the second row the contribution of each fluid (radiation and matter) to the total δ_M is represented.

We will finish this subsection with a critical discussion about the role baryons play in structure formation and in the evolution of perturbations. Along our derivations, we have considered that matter and radiation do not interact[4] apart from the metric perturbations each one induces, considering that his perturbations have been caused by the density perturbations of the dominant fluid in each epoch. This is not an illogical supposition, as 84.2% of the matter content of the Universe consists in dark (non-baryonic) matter, which does not interact with electromagnetic radiation. In spite of this, visible astronomical features, such as stars or galaxies are composed by baryonic mater (15.8% if the matter content of the Universe), so it is clear that baryons play a significant role. While the extensions constraints in our work prevent us from derivating the evolution of the baryonic matter perturbations in full detail and rigor, a small discussion is necessary.

During the Radiation Dominated Era and the Matter Dominated one until Recombination ($a_{recomb} = 9.08 \cdot 10^{-4}$), photons and electrons are tightly coupled via Thomson scattering, and in turn electrons and protons interact via Coulomb scattering, acting as a single fluid. As the radiation density was much greater than the baryon one, we can consider this fluid to behave as a radiation dominated one, with its fluctuations behaving as such (see Section 4.2), at least for small scales (recall that during the Radiation Dominated Era perturbation modes with $k < \frac{\pi}{2}(a_r - a_{i*})^{-1}\sqrt{3\Omega_r}H_0 a_0^2 c^{-1} \approx 0.02$ Mpc^{-1} do not experience oscillations). Meanwhile, dark matter is not coupled to radiation, so once its density is large enough (matter-radiation equality, a_r), $\delta_{M,c}$ grows as $\mathcal{O}(a)$. While dark matter does not interact with radiation through electromagnetic processes, it nonetheless has important effects. When light from the CMB travels towards Earth, where it is detected, it must go through regions where dark matter inhomogeneities are present, coming in and out from the gravitational waves, where it can be redshifted (cooled). Due to this phenomenon (as well as others) the measured CMB power spectrum does not directly correspond

[4]Under some theories discussing the hypothetical nature of dark matter, there exists the possibility that non-baryonic matter is susceptible to weak interaction. Under this supposition, during the early stages of the Radiation Dominated Era the neutrino (which behave as radiation) density could be so high that dark matter was coupled to radiation. At some point during the progressive expansion and cooling of the Universe, the neutrino density was low enough to decouple, giving way to the hypothetical *Cosmic Neutrino Background*, with an expected temperature of $T \approx 1.945$ K, and which has not been directly detected yet. This speculative coupling between dark matter and neutrinos would have left an imprint on the CNB signal. A small discussion on this subject can be found in Section 3.6 from [**Piattella, 2018**].

with the radiation density perturbations power spectrum when it was emitted. This way it is necessary to work with the so-called *transference functions*, for which we recommend the reader Chapter 10 from [**Piattella, 2018**].

As radiation temperature and density keep decaying, electrons are able to combine with protons to form neutral hydrogen, allowing photons to travel freely and decouple from baryons. Baryons then start falling into the gravitational potentials caused by dark matter density perturbations. While baryon density perturbations $\delta_{M,b}$ end up converging to those from dark matter, $\delta_{M,c}$, the different initial values for $\delta_{M,b}$ and $\gamma_{M,b}$ coming from the radiation density oscillations before decoupling have a small, but noticeable effect, resulting in what are called *baryon acoustic oscillations (BAO)*, on the distribution of baryonic matter (galaxy clusters). For the formalism behind the qualitative relation between dark matter and baryons, and the formation of BAOs, we recommend Section 9.2 from [**Piattella, 2018**].

4.5.3. Comparison between complete evolution and growing modes-only solution. In the final section of this chapter we will assess the differences between considering a complete evolution, using the all the terms for evolution equations developed in this chapter, and considering only the dominant, growing modes. This way we will evaluate the importance of considering the subdominant and decaying modes, and how they serve as corrections for the dominant terms.

While the simplifications on the Matter Dominated equation are immediate, for the Radiation Dominated Era the auxiliary functions have the following leading terms:

$$\alpha(z) \approx \frac{1}{zz_i^2}, \qquad \beta(z) \approx 1, \qquad \lambda(z) \approx -\frac{1}{z_i^2}, \qquad \gamma(z) \approx -\frac{1}{z_i} \tag{4.5.5a}$$

$$a(z) \approx \frac{1}{zz_i^2}, \qquad b(z) \approx 1, \qquad c(z) \approx -\frac{1}{z^2 z_i^2}, \qquad d(z) \approx \frac{1}{z_i} \tag{4.5.5b}$$

This way, we can approximate the perturbations evolution by their leading terms:

$$\hat{\delta}_M(a,k) \approx \begin{cases} \left[\frac{\hat{\delta}_{M,i}(k)}{zz_i^2} + \frac{\hat{\gamma}_{M,i}(k)}{z_i}\right]\sin(z - z_i) - \frac{1}{z_i^2}\left[\hat{\delta}_{M,i}(k) + \hat{\gamma}_{M,i}(k)\right]\cos(z - z_i) & a \in [a_p, a_r] \\ \frac{1}{5}\frac{a}{a_r}\left[3\hat{\delta}_{M,i}k) + 3\hat{\gamma}_{M,i}(k)\right] & a \in [a_r, a_m] \end{cases} \tag{4.5.6a}$$

$$\hat{\gamma}_M(a,k) \approx \begin{cases} \left[\frac{\hat{\delta}_{M,i}(k)}{z_i^2} + \frac{z}{z_i}\hat{\gamma}_{M,i}(k)\right]\cos(z - z_i) + \left[-\frac{1}{zz_i^2}\hat{\delta}_{M,i}(k) + \frac{z}{z_i}\hat{\gamma}_{M,i}(k)\right]\sin(z - z_i) & a \in [a_i, a_r] \\ \frac{1}{5}\frac{a}{a_r}\left[3\hat{\delta}_{M,i})k) + 3\hat{\gamma}_{M,i}(k)\right] & a \in [a_r, a_m] \end{cases} \tag{4.5.6b}$$

where $z = \frac{ak}{\sqrt{3\Omega_r} H_0 a_0^2}$ in the same way as before. The expression of the power coefficients $\Delta^2_{\delta_M}$, $\Delta^2_{\gamma_M}$ and Ξ under this consideration is immediate and we will refrain from writing it here as it is not our objective to analyze it.

The evolution given by the leading terms for the density and power perturbations power spectrum coefficients for several k modes is presented in Figure 12, as well as the relative difference between them and the complete evolution (see Figure 9),

- First of all, regarding the evolution of the coefficients, it is clear that in both cases the overall behavior is reproduced, namely the superhorizon evolution of the coefficients and the oscillations present inside the horizon in the radiation era. As it can be seen in the second plot and in Figure 12, though the values obtained at the end of the Radiation Dominated period are approximately of the same order of magnitude than those obtained using the complete evolution, for $a \ll a_r$ the power coefficients for the density perturbations are many orders of magnitude larger, progressively converging for larger a. This does not happen for the power perturbations, where the difference is much smaller in the superhorizon regime of the radiation era (around 15% for all modes, probably due to depreciated constant terms). As given by the multiple "spikes" in the oscillatory parts, it is evident that the perturbations given by the complete evolution and those by the dominant terms have a small phase mismatch in this phenomenon, which can affect int he initial conditions for the following era (see fourth plot of the figure). Another side effect of only considering the dominant terms of the evolution expressions is that continuity at the transition from radiation to matter dominated is lost.

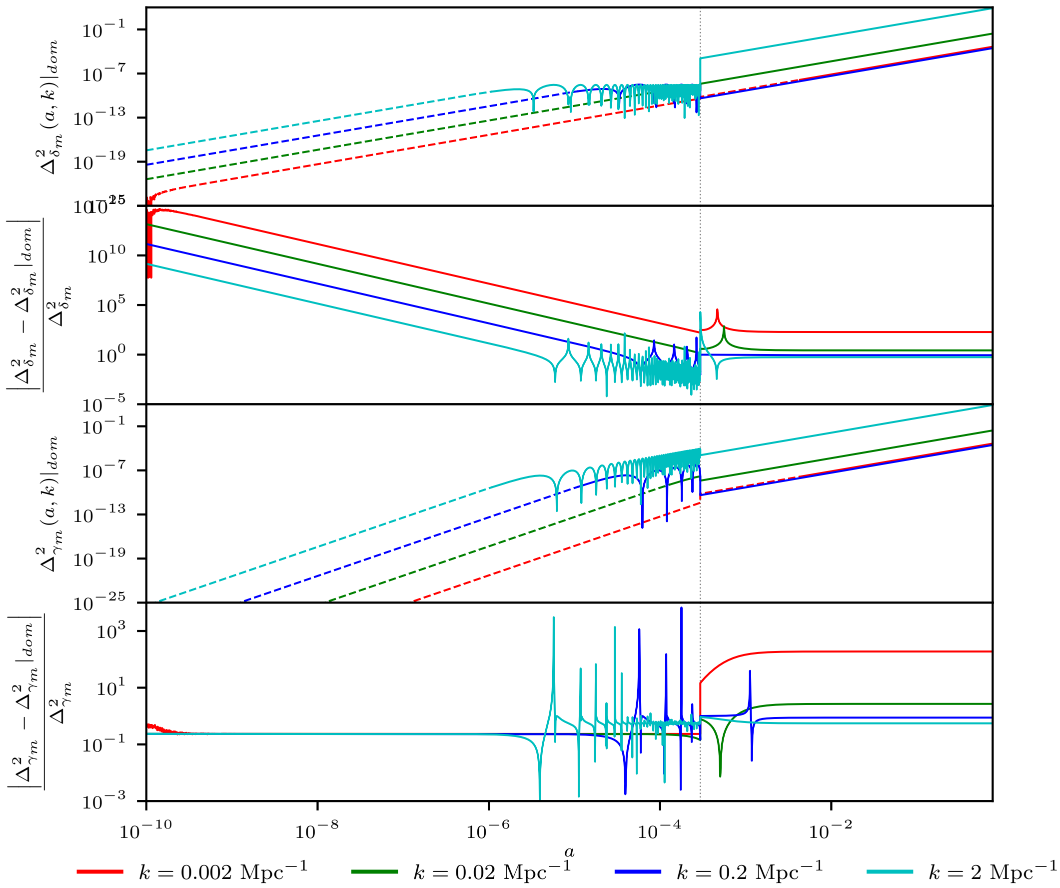

FIGURE 12. Evolution for several k modes of the power coefficients of density perturbations using only the dominant modes, with the evolution of the perturbations while outside the horizon is given in dashed lines. The limits between each of the epochs is represented by vertical, dotted lines. The second and fourth plots show the relative difference between the complete evolution and only the leading one for density and power perturbations.

- Regarding specifically the relative differences, we have that they seem to be larger for modes with smaller k. As it has been already said, the discrepancies between the two approaches to the evolution of the density perturbations (complete versus dominant) are more important in the earlier stages of the Radiation Dominated Era, when the auxiliary functions $\{\alpha(z), ..., d(z)\}$ could not be accurately approximated by their leading terms, as the not dominant modes are not yet negligible. For the Matter Dominated Era these discrepancies are much smaller (but still of several orders of magnitude), as shortly after the begin of this era only the growing terms are relevant. However, as the initial conditions at a_r are different in each of the approaches, the coefficients end up featuring a constant difference between the two cases.

This means that, while the overall evolution is similar for the complete expression and that only featuring dominant terms, lower order and decaying terms are needed to ensure a continuous evolution and to obtain the right final conditions at the end of the Radiation Era which serve as initial conditions for the following. Discrepancies are larger for density perturbations during the Radiation Dominated Era, and for power perturbations during the Matter Dominated one. For both cases the final discrepancies are larger for smaller k, for which it is clear that the remaining terms from the complete evolution expressions are quite relevant.

CHAPTER 5

Conclusions and future work

Throughout this work, we have obtained the complete expressions describing the evolution of the density and power perturbations, $\delta_M(a,k)$ and $\gamma_M(a,k)$, in terms of the scale factor a and Fourier mode k, as well as the initial values of the perturbations. Our results have been obtained solving Bardeen's Evolution Equation (eq. 3.2.26) for the radiation and matter dominated cases, agreeing with the standard approach taken when obtaining analytic solutions for the perturbations evolution.

We have reviewed the linear perturbations formalism for the metric $g_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$, as well as the concepts of gauge-invariant magnitudes and gauge choice. In our way to a gauge-invariant evolution equation for scalar perturbations completely written in terms of components of the stress-energy tensor, we have carried out all the intermediate steps and calculations Bardeen omitted in [**Bardeen, 1980**], as well as updating the used notation and offering physical interpretation to the series of magnitudes and expressions obtained.

Under important but realistic simplifications (namely working under a flat Universe evolving as if it was composed by a single perfect fluid, without anisotropic stress and with initial adiabatic fluctuations), we have obtained evolution equations for δ_M and γ_M. Our approach and results are original in that not only they offer the complete evolution of each k mode, including decaying terms or lower order corrections, but they also express this evolution in terms of the initial values for δ_M and γ_M. As the evolution equations being solved are second order ODEs (after the Fourier transform has been applied to the original PDE so that we are able to work with individual k modes), in order to completely determine how perturbations evolve, the initial value of the first derivative is also needed. While traditional approaches implicitly assume $\delta_M \sim \gamma_M$, we have shown that in some situations, such as in the end of the Radiation Dominated Era, this is not the case. As it is evident from Figure 12, while the overall behavior is given by the dominant terms, the exclusion of decaying and lower order terms gives way to important discrepancies with the perturbation evolution obtained using the complete expression, specially for earlier times.

Specifically, we have solved the evolution equations for $\omega = \frac{1}{3}$ (radiation dominated) and $\omega = 0$ (matter dominated), with the obtained expressions agreeing with the traditional approaches, while adding important corrections. Another strength of our solutions is that they are valid for all scales (both inside and outside the horizon), while usually the evolution of sub- and super-horizon perturbations is obtained separately after carrying out different simplification. This allows for a smooth transition as the perturbations cross the horizon, rather than having to "glue" the different expressions, as the traditional approaches when finding analytic solutions do (Chapter 5 from [**Baumann, 2016**]).

On the negative side, though at first sight there seems not to be any indication that eq. 3.2.26 is not valid for a $\omega = -1$ dominated Universe (such as an inflationary or Λ-dominated one), the obtained solutions do not agree with the qualitative discussion we made in Chapter 3 about the behavior of perturbations on superhorizon scales, nor with the current, observed evolution of perturbations or bibliographical sources. We conclude that, as in the derivation of eq. 3.2.26 we have assumed implicitly that $\omega \neq -1$, a different expression, probably considering non-adiabatic or anisotropic terms, is needed in order to obtain the evolution of perturbations in this case.

Anyhow, our work allows us to work with the standard initial conditions coming from the curvature perturbation spectrum $\Delta^2_{\mathcal{R}}(k)$, and the complete evolution of the different power spectra $\Delta^2_{\delta_M}$, $\Delta^2_{\gamma_M}$ and Ξ , of considerable statistical importance, during the Radiation and Matter Dominated Eras, where important events, such as Recombination and the first stages of large scale structure formation, occur. Furthermore, our analytic solutions allow us to obtain expressions for specific events, such as the scale factor at which the perturbations modes start an oscillating behavior during the Radiation Dominated Era, or the different contributions the initial conditions have in the evolution of the perturbations.

Regarding the future development of this work, there are several directions of interest we have not been able to develop due to time and space constraints. Some of the future paths that can follow this work are the following:

- First of all, it is important to obtain an evolution equation analogous to eq. 3.2.26 for the $\omega = -1$ case, and use it to obtain solutions which describe the evolution of the perturbations, checking that they agree with the bibliographical, observational and qualitative properties mentioned in the text, such as evolving as $\mathcal{O}(a^{3\omega_{eff}})$ for superhorizon scales or the matter density perturbations being constant during the Λ Dominated Era. The relation of these solutions with those with $\omega \to -1$ would also be interesting, in order to study what makes this evolution special, or if there can be a continuous, dynamical evolution in ω, as some models have proposed (read [**Vikman, 2005**]).

- Regarding the evolution of perturbations during inflation, the standard approach consists in taking the curvature fluctuations $\mathcal{R}$, which are frozen once they cross the horizon, and using them as sources for the initial density perturbations. However, as during inflation the Universe expands exponentially, as governed by a $\omega = -1$ equation of state, we could consider initial density and power fluctuations during inflation, which then evolve as given by the hypothetical evolution equation for $\omega = -1$. To do so we would need to work with a specific inflaton potential $V(\phi)$ from which obtaining the associated stress-energy tensor $T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta} - V(\phi)\right)$ (which would directly depend on the shape of $V(\phi)$), the fluctuations of which could be of quantum nature, and feature anisotropic and non-adiabatic terms.
- One important feature when solving the $\omega = 0$, corresponding to a matter dominated Universe in which its components (such as galaxies) did not interact, is that the ∇^2 (or $-k^2$) term from eq. 3.2.26 vanished, meaning that the evolution was scale-invariant (and as a result the evolution equation is quite easy to solve). However, under real conditions, different galaxies do interact, if quite weakly. The *structural stability* of the solutions should be studied, for a $\omega = 0 + \varepsilon$ equation of space (with $|\varepsilon| \ll 1$). The scale-invariant evolution should be recovered for a $\varepsilon \to 0$ limit.
- We could try to study baryons (see Subsection 4.5.2) within our solutions. This adds the complication of separating the evolution of dark matter (which does not interact with radiation other than gravitationally by means of the metric perturbations) and baryonic matter, which is coupled to radiation until Recombination. This approach would probably require additional considerations, be it via a modification of eq. 3.2.26, which does not take into account electromagnetic interactions such as Thomson or Coulomb scattering, or through external conditions and approximation in the relevant moments (which otherwise would probably difficult the expression of the perturbation evolution via analytic expressions). Neutrinos could also be studied, though the effect of taking them into consideration would be probably small, and the additional terms involved would complicate the resolution of the equations.
- One of our (implicit) assumptions when solving the evolution equations was that the fluid, with equation of state ω, experiencing the perturbations was the same dominating the evolution of the Universe along a certain epoch. In the $\omega = -1$ case we discussed the possibility that the fluid experiencing the perturbations (ω_{eff}) had not the same equation of state as that dominating the Universe expansion (ω). It would be an interesting (and not particularly difficult) exercise to repeat this for the radiation and matter dominated era. Specially interesting is the case of the dark matter, which, though at first being driven by the metric perturbations coming from radiation inhomogeneities, could then start evolving "independently", maybe even before the radiation dominated epoch has finished.
- While observational results impose important constraints on initial perturbation modes different from adiabatic ones (read Section 9 from [**Planck Col.-Infl., 2018**]), from a theoretical point of view it would be interesting to study the evolution of perturbations with $\eta \neq 0$. If we consider the entropy perturbations to be constant with time, then it is necessary to add a $-k^2(\bar{P}a^3\eta)$ to the right side eq. 3.2.26, making the ODE non-homogeneous.
- Finally, we could try to obtain the theoretical shape of different cosmological observables, such as the anisotropies of the CMB or the matter distribution spectrum. This would require introducing the theory behind transference functions (read Chapters 9 and 10 from [**Piattella, 2018**]) and the use of computational programs such as CAMB ([**CAMB, 2014**]). However, these programs usually use the curvature power spectrum $\Delta^2_{\mathcal{R}}(k)$ rather than the scalar perturbations ones as input. A modification of these codes which allowed to use $\Delta^2_{\delta_M}$, $\Delta^2_{\gamma_M}$ and Ξ as input would have to be implemented.

In conclusion, our results are valuable and important when trying to obtain exact and complete solutions for the evolution of cosmological perturbations, and they open an important number of roads which could be followed in the future.

APPENDIX A

Some notes on Fourier Space. Statistics.

In this appendix we will recall the basic tools of Fourier Analysis and Statistics used along the text.

A.1. Basic notions of the Fourier Transform

Given an arbitrary, no necessarily continuous, square-integrable complex valued function over $\mathbb{R}^n$, $f \in L^2(\mathbb{R}^n;\mathbb{C})$, that is, with $\int_{\mathbb{R}^n} |f(\vec{x})|^2 \mathrm{d}^n x < +\infty$, we define its *Fourier Transform* as

$$\mathcal{F}(f)(\vec{k}) \equiv \hat{f}(\vec{k}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \mathrm{d}^n x, \tag{A.1.1}$$

and analogously the *Inverse Fourier Transform* as

$$\mathcal{F}^{-1}(\hat{f})(\vec{x}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \mathrm{d}^n k \tag{A.1.2}$$

The *Fourier Inversion Theorem* states that it is possible to recover the original function from its transformed counterpart by means of the Inverse Fourier Transform:

$$f(\vec{x}) = \mathcal{F}^{-1}\left(\mathcal{F}(f)\right)(\vec{x}) \quad \forall \vec{x} \in \mathbb{R}^n \tag{A.1.3}$$

By the way they are defined, it is immediate that both the "Direct" and Inverse Fourier Transform are linear operators. It is possible to interpret the Fourier Transform as a generalization of the *Fourier Series* for non-periodical functions. Whereas we can decompose any periodic function f with a period with length T as an infinite sum of plane waves with "quantized" wavenumber:

$$f(x) = \sum_{n\in\mathbb{Z}} f_n e^{ik_n x}, \qquad \text{with} \quad f_n = \frac{1}{T}\int_0^T f(x) e^{-ik_n x} \mathrm{d}x \quad \text{and} \quad k_n = \frac{2\pi n}{T}, \tag{A.1.4}$$

in the non-periodical time, this time not restricted to the real line, the function is decomposed as superposition of plane waves with continuous wavenumber $\vec{k} \in \mathbb{R}^n$:

$$f(\vec{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \mathrm{d}^n k, \qquad \text{with} \quad \hat{f}(\vec{k}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \mathrm{d}^n x \tag{A.1.5}$$

Whereas in other fields of Physics the wavenumber $\vec{k}$, which has units of inverse length, has a direct interpretation, such as $\hbar\vec{k}$ being the linear momentum in Quantum Mechanics, here $\vec{k}$ will simply allow us a simpler treatment of the solutions of certain differential equations in terms of plane waves, with $\hat{K} = \frac{\vec{k}}{\|\vec{k}\|}$ being its orthogonal direction and $\|\vec{k}\|^{-1}$ its "physical wavelength".
As just mentioned, we will use the Fourier Transform in order to express the solutions of different partial differential equations which feature the Laplacian ∇^2 of scalar functions. As ∇^2 involves second partial spatial derivatives, its expression can be quite complicated. However, it is easy to see that, given any function $f \in L^2(\mathbb{R}^n;\mathbb{C})$ and taking the Fourier Transform of $\nabla^2 f$,

$$\begin{aligned}
\mathcal{F}\left(\nabla^2 f\right)(\vec{k}) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \nabla^2 f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \mathrm{d}^n x = \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 f(\vec{x})}{\partial x_j^2} e^{-i\vec{k}\cdot\vec{x}} \mathrm{d}^n x \\
&= -\frac{1}{(2\pi)^{n/2}} \sum_{j=1}^n \int_{\mathbb{R}^n} \frac{\partial f(\vec{x})}{\partial x_j} \frac{\partial}{\partial x_j} e^{-i\vec{k}\cdot\vec{x}} \mathrm{d}^n x = \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^n \int_{\mathbb{R}^n} f(\vec{x}) \frac{\partial^2}{\partial x_j^2} e^{-i\vec{k}\cdot\vec{x}} \mathrm{d}^n x \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\vec{x}) \nabla^2 e^{-i\vec{k}\cdot\vec{x}} \mathrm{d}^n x = -k^2 \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \mathrm{d}^n x \\
&= -k^2 \mathcal{F}(f)(\vec{k}),
\end{aligned} \tag{A.1.6}$$

where $k \equiv \|\vec{k}\|$ and we have used that both f and its derivatives vanish as $\|\vec{x}\| \to \infty$. Using this and the linearity of the Fourier Transform, we can convert a single PDE in terms of time and spatial (through ∇^2) derivatives in the "real space", to a continuum of ODEs in terms of time derivatives, where $k \in \mathbb{R}$ appears as a constant identifying the different solutions in the "Fourier Space".

In the case we are working on a (pseudo-)Riemannian manifold, the definition of the Fourier Transform gets trickier, as the function decompositions are not given in terms of the exponentials $\{e^{i\vec{k}\cdot\vec{x}}\}_{\vec{k}\in\mathbb{R}^n}$, but rather in in terms of the eigenfunctions $\left\{E_{\vec{k}}(\vec{x})\right\}_{\vec{k}\in\mathbb{R}^n}$ of the *Laplace–Beltrami operator* Δ, the analogue of the Laplacian in more general manifolds with an attached metric $g_{\mu\nu}$ (for more on this, read chapter 15 from [**Lee, 2003**]), expressed in local coordinates as

$$\Delta f := \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|\det g|} g^{ij} \frac{\partial}{\partial x^j} f \right) \tag{A.1.7}$$

As one could imagine, this evolves into quite complicated expressions when trying to do concrete calculations. We are, however, still able to go from "real" to Fourier space in order to simplify the PDEs involved and obtain solutions in terms of the Fourier modes $\vec{k}$, by replacing the ∇^2 or Δ operator by $-k^2$ and continue working in the Fourier space.

A.2. Correlation Function and Power Spectrum

We will now give some basic definition and results regarding some of the statistics used along the text.

We first consider an homogeneous, isotropic, random field f taking real values. We define the *two-point correlation function*, ξ_f as

$$\xi_f(r) := \langle f(\vec{x}) f(\vec{x} - \vec{r}) \rangle = \int f(\vec{x}) f(\vec{x} - \vec{r}) \mathrm{d}^3 x, \tag{A.2.1}$$

where $r = \|\vec{r}\|$ gives information only about the distance between any two points, as we have assumed isotropy (so the direction is not important) and homogeneity (so that the precise two points are not needed).

We can now try to do the same for the Fourier transform of f, which might be a complex function. As $f(\vec{x})$ is real for all $\vec{x}$, it is immediate that $\hat{f}(\vec{k})^* = \hat{f}(-\vec{k})$. Now,

$$\begin{aligned} \xi_{\hat{f}}(\rho) &= \langle \hat{f}(\vec{k}) \hat{f}(\vec{k} - \vec{\rho})^* \rangle = \langle \hat{f}(\vec{k}) \hat{f}(-\vec{k} + \vec{\rho}) \rangle = \frac{1}{(2\pi)^{3/2}} \iiint f(\vec{x}) e^{-i\vec{x}\cdot\vec{k}} f(\vec{x}') e^{i\vec{x}'\cdot(\vec{k}-\vec{\rho})} \mathrm{d}^3 k \mathrm{d}^3 x \mathrm{d}^3 x' \\ &= \frac{1}{(2\pi)^3/2} \iiint f(\vec{x}) f(\vec{x} - \vec{r}) e^{-i\vec{r}\cdot(\vec{k}-\vec{\rho})} e^{-i\vec{\rho}\cdot\vec{x}} \mathrm{d}^3 k \mathrm{d}^3 x \mathrm{d}^3 r \end{aligned} \tag{A.2.2}$$

where we have used $\vec{x} = \vec{x}' - \vec{r}$ and the definition of the Dirac delta in $\mathbb{R}^n$ as

$$\delta(\vec{\alpha}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\vec{x}\cdot\vec{\alpha}} \mathrm{d}^n x \tag{A.2.3}$$

We can thus define the *power spectrum* associated to f as the function $P_f(\vec{k})$ such that

$$\langle \hat{f}(\vec{k}) \hat{f}(\vec{k}')^* \rangle = \frac{1}{(2\pi)^{3/2}} \delta(\vec{k} - \vec{k}') P_f(\vec{k}), \tag{A.2.4}$$

and an associated *dimensionless power spectrum*

$$\Delta_f^2(\vec{k}) = \frac{k^3}{2\pi^2} P_f(\vec{k}) \tag{A.2.5}$$

It is easy to see that both $P_f(\vec{k})$ and $\Delta_f^2(\vec{k})$ can be obtained directly from $\hat{f}(\vec{k})$.
Finally we present how the two-point correlation function in the real space can be obtained from $\Delta_f^2(\vec{k})$, under homogeneity and isotropy conditions:

$$\boxed{\xi_f(r) = \int_0^\infty \Delta_f^2(\vec{k}) \mathrm{sinc}(kr) \mathrm{d} \ln k} \tag{A.2.6}$$

Proof. We will use the fact that f takes real values, so $f = f^*$. Now, by the definition of the two-point correlation function,

$$\xi_f(r) = \langle f(\vec{x}) f(\vec{x} - \vec{r}) \rangle = \int f(\vec{x}) f(\vec{x} - \vec{r}) \mathrm{d}^3 x = \int f(\vec{x}) f(\vec{x} - \vec{r})^* \mathrm{d}^3 x \tag{A.2.7}$$

Using now the Inverse Fourier Transform and the definition of Dirac delta,

$$\begin{aligned} \xi_f(r) &= \frac{1}{(2\pi)^3} \iiint \hat{f}(\vec{k}) e^{i\vec{x}\cdot\vec{k}} \hat{f}(\vec{k}')^* e^{-i(\vec{x}-\vec{r})\cdot\vec{k}'} \mathrm{d}^3 k \mathrm{d}^3 k' \mathrm{d}^3 x = \frac{1}{(2\pi)^3} \iiint e^{i\vec{x}\cdot(\vec{k}-\vec{k}')} \hat{f}(\vec{k}) \hat{f}(\vec{k}')^* e^{i\vec{r}\cdot\vec{k}} \mathrm{d}^3 k \mathrm{d}^3 k' \mathrm{d}^3 x \\ &= \frac{1}{(2\pi)^{3/2}} \iint \delta(\vec{k} - \vec{k}') \hat{f}(\vec{k}) \hat{f}(\vec{k}')^* e^{i\vec{r}\cdot\vec{k}} \mathrm{d}^3 k \mathrm{d}^3 k' = \frac{1}{(2\pi)^{3/2}} \int |\hat{f}(\vec{k})|^2 e^{i\vec{r}\cdot\vec{k}} \mathrm{d}^3 k \end{aligned} \tag{A.2.8}$$

Due to the isotropy and homogeneity of f (and thus of $\hat{f}$), $|\hat{f}(\vec{k})|^2$ should depend only on $k = \|\vec{k}\|$, rather than its direction. This way, we can obtain its value by integrating over all the possible values of the two-point correlation function:

$$|\hat{f}(\vec{k})|^2 = \int \xi_{\hat{f}}(|\vec{k} - \vec{k}'|)\mathrm{d}^3k' = \frac{1}{(2\pi)^{3/2}} \int \delta(\vec{k} - \vec{k}')P_f(\vec{k})\mathrm{d}^3k' = \frac{1}{(2\pi)^{3/2}} P_f(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \frac{2\pi^2}{k^3} \Delta_f^2(\vec{k}) \tag{A.2.9}$$

Assuming now the x-axis is aligned with $\vec{r}$, we have that, in spherical coordinates,

$$\xi_f(r) = \frac{1}{4\pi} \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{1}{k} P_f(k) e^{-ikr\sin\theta\cos\phi} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}k = \int_0^\infty P_f(k)\mathrm{sinc}(kr)\mathrm{d}\ln k \tag{A.2.10}$$

where we have used $\int_0^{2\pi} \int_0^\pi e^{-i\alpha\sin\theta\cos\phi} \sin\theta \mathrm{d}\theta \mathrm{d}\phi = 4\pi\mathrm{sinc}(\alpha)$. □

It is immediate that the variance of f is given then by

$$\langle f^2 \rangle = \xi_f(0) = \int_0^\infty \Delta_f^2(k)\mathrm{d}\ln k \tag{A.2.11}$$

APPENDIX B

Some notes on (pseudo-)Riemannian Geometry. Calculations for chapter 3

In this appendix we will introduce the necessary concepts on (pseudo-)Riemannian Geometry needed for Chapters 2 and 3.

B.1. Basic concepts of differential manifolds

We will begin with the definition of manifold, charts and coordinates.

Definition 1. *Let M be an arbitrary set*

- *An one-to-one map $x : U \subseteq M \to \mathbb{R}^n$ such that $x(U)$ is an open set[1] of $\mathbb{R}^n$ is called* chart. *The different components $x = (x^1, ..., x^n)$ are called* coordinate functions.
- *An* atlas *is a set of charts $\mathcal{A} = \{(x_\alpha, U_\alpha)\}_\alpha$ such that their domains cover all M, $\cup_\alpha U_\alpha = M$. An atlas is said to be* differentiable *if for every two charts (x, U) and (y, V) such that $U \cap V \neq \emptyset$, the function $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ is a diffeomorphism (this is, a bijective, infinitely differentiable function such that its inverse function is also bijective and infinitely differentiable).*
- *We will call* differential manifold *a set M with an associated differentiable atlas $\mathcal{A}$. If the maps from $\mathcal{A}$ go to $\mathbb{R}^n$, M is said to have dimension n.*

Different atlas $\mathcal{A}$ and $\mathcal{A}'$ can induce the same differential structure on M if their union $\mathcal{A} \cup \mathcal{A}'$ is itself a differentiable atlas. A more rigorous definition uses a maximal atlas, $\mathcal{A}^+$, such that it cannot be included in a bigger one. However, as it can be shown that for every atlas there exist a maximal atlas containing it, our definition is still valid.

We will now introduce a few differential concepts which generalize the vector calculus from $\mathbb{R}^n$ to arbitrary differential manifolds.

Definition 2. *Let M be a n-dimensional differential manifold.*

- *Given a point $p \in M$, the* tangent space of M at p, *T_pM is defined as the set of all the possible tangent vectors at p defined by the curves in M going through p. It can be shown that T_pM is a n-dimensional vector space over $\mathbb{R}$, with the following basis*

$$\mathcal{B}_p = \left\{ \left(\frac{\partial}{\partial x^i} \right)_p \right\}_{i=1}^n , \tag{B.1.1}$$

  in the sense that they can be thought of as differential operators.
- *We define the* cotangent space of M at p, *T_p^*M, as the dual space of T_pM, that is, another n-dimensional vector space over $\mathbb{R}$, with a basis $\mathcal{B}_p^* = \{(\mathrm{d}x^i)_p\}_{i=1}^n$. The elements of T_p^*M are interpreted as linear maps from T_pM to $\mathbb{R}$, acting as*

$$\mathrm{d}x^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j = \begin{cases} 1 & \textit{if} \quad i = j \\ 0 & \textit{if} \quad i \neq j \end{cases} \tag{B.1.2}$$

  The same way, T_pM can be thought as the dual space of T_p^*M, with its elements acting as linear maps from T_p^*M to $\mathbb{R}$:

$$\frac{\partial}{\partial x^i}(\mathrm{d}x^j) = \delta^j_i = \begin{cases} 1 & \textit{if} \quad i = j \\ 0 & \textit{if} \quad i \neq j \end{cases} \tag{B.1.3}$$

- *The (disjointed) unions of all the tangent and cotangent spaces are respectively called* tangent *and* cotangent *bundles:*

$$TM = \bigcup_{p\in M} T_pM, \qquad T^*M = \bigcup_{p\in M} T_p^*M \tag{B.1.4}$$

  It can be shown that both TM and T^*M are differentiable manifolds of dimension $2n$.
- *Elements from TM and T^*M are respectively called contravariant and covariant vectors. Given $v \in TM$, $\alpha \in T^*M$, there will exist unique sets $\{v^i\}_{i=1}^n$ and $\{\alpha_i\}_{i=1}^n$ of coefficients, such that*

$$v = v^i \frac{\partial}{\partial x^i}, \qquad \alpha = \alpha_i \mathrm{d}x^i, \tag{B.1.5}$$

  where the Einstein's summation convention has been used. As every point $p \in M$ has at least a chart (x, U) on such that $p \in U$, $\frac{\partial}{\partial x^i}$ and $\mathrm{d}x^i$ vary smoothly over M. The change from one chart to the another is given as follows:

[1] While we will not focus on topological aspects, an open subset of $\mathbb{R}^n$ can intuitively thought about as the union of an arbitrary number of open balls of the type $B(x_0, \epsilon) := \{p = (p^1, ..., p^n) \in \mathbb{R}^n : \|p\| < \epsilon\}$.

- *Given two charts (x,U) and (x',V') such that $U\cap U'\neq\emptyset$, the coordinate change from one chart to the other on $U\cap U'$ is easily performed by*

$$v=v^i\frac{\partial}{\partial x^i}=\bar{v}^i\frac{\partial}{\partial \bar{x}^i},\quad \text{with}\quad \bar{v}^i=v^j\frac{\partial \bar{x}^i}{\partial x^j},\qquad \alpha=\alpha_i\mathrm{d}x^i=\bar{\alpha}_i\mathrm{d}\bar{x}^i,\quad \text{with}\quad \bar{\alpha}_i=\alpha_j\frac{\partial x^j}{\partial \bar{x}^i} \tag{B.1.6}$$

A fundamental concept in Differential Geometry is that of *tensor*. First of all we will introduce the *tensor product*:

Definition 3. *Let U and V be two finite dimensional vector spaces over $\mathbb{R}$.*

- *Consider the vector space $\langle U\times V\rangle$ generated by all the linear combinations of elements from $U\times V$. We will consider the following operation $\otimes: U\times V\to\langle U\times V\rangle$, called* tensor product, *such that*

$$\begin{gathered}(v_1+v_2)\otimes w=v_1\otimes w+v_2\otimes w\\ v\otimes(w_1+w_2)=v\otimes w_1+\otimes w_2\\ (\lambda v)\otimes v=\lambda(v\otimes w)\\ v\otimes(\lambda w)=\lambda(v\otimes w),\end{gathered} \tag{B.1.7}$$

  with $v,v_1,v_2\in U$, $w,w_1,w_2\in V$ and $\lambda\in\mathbb{R}$. The space defined by this relation is called tensor space, *$U\otimes V$.*
- *Let V^* be the dual space of V (in the same sense as T_pM and T_p^*M, the set of linear maps from V to $\mathbb{R}$), and vectors $t_1\in V$, $t_2\in V^*$. Given $u\in V^*$, $w\in V$, $t_1\otimes t_2$ acts on these elements as*

$$t_1\otimes t_2(u,v)=t_1(u)t_2(v)\in\mathbb{R} \tag{B.1.8}$$

  It is easy to see that under this behavior, the tensor product is commutative, and $V\otimes V^*$ is immediately identifiable with $V^*\otimes V$. We can generalize this to a finite number of products. Let $\{e_1,...,e_n\}$ be a basis of V and $\{e^1,...,e^n\}$ the dual basis of V^*, in the sense that $e^i(e_j)=e_j(e^i)=\delta^i_j$. We will define the tensors of (r,s) type over V, *$\mathfrak{T}^r_s(V)$, as the n^{r+s} dimensional vector space generated by*

$$\mathcal{B}_{r,s}:=\{e_{i_1}\otimes\overset{r}{\cdots}\otimes e_{i_r}\otimes e^{j_1}\otimes\overset{s}{\cdots}\otimes e^{i_s}: i_k,j_k\in\{1,...,n\}\}$$

  Using Einstein's summation convention, the elements of $\mathfrak{T}^r_s(V)$ can be expressed as

$$t=t^{i_1,...,i_r}_{j_1,...,j_s}e_{i_1}\otimes\overset{r}{\cdots}\otimes e_{i_r}\otimes e^{j_1}\otimes\overset{s}{\cdots}\otimes e^{j_s} \tag{B.1.9}$$

- *Given $t\in\mathfrak{T}^r_s(V)$ and $t'\in\mathfrak{T}^{r'}_{s'}(V)$, $t\otimes t'$ is a $(r+r',s+s')$ tensor.*
- *Given a n-dimensional differential manifold M, we can generalize these concepts, identifying $V=T_pM$ and $T_p^*M=V^*$ for a given $p\in M$. As the basis elements $\frac{\partial}{\partial x^i}$ and $\mathrm{d}x^i$ vary smoothly over the domain of each chart, we can define the* (r,s)-tensor field over M *as*

$$\mathfrak{T}^r_s(M)=\left\{f^{i_1,...,i_r}_{j_1,...,j_s}\frac{\partial}{\partial x^{i_1}}\otimes...\otimes\frac{\partial}{\partial x^{i_r}}\otimes\mathrm{d}x^{j_1}\otimes...\otimes\mathrm{d}x^{j_s}: i_k,j_k\in\{1,...,n\}, f^{i_1,...,i_r}_{j_1,...,j_s}\in\mathfrak{F}(M)\right\}, \tag{B.1.10}$$

  where $\mathfrak{F}(M)$ is the set of infinitely differentiable functions from M to $\mathbb{R}$. It is understood that the elements from $\mathfrak{T}^r_s(M)$ act over each point. When the coordinate basis is implicit, we will denote tensors by their set of coefficients $f^{i_1,...,i_r}_{j_1,...,j_s}$.
  We will call the $(0,r)$ tensors covariant tensors, *$(0,s)$ tensors* contravariant tensors, *and (r,s) tensors* mixed tensors. *In particular, $(0,1)$ tensors are called* vector fields, *$\mathfrak{X}(M)$.*

- *Given two charts (x,U) and (x',V') such that $U\cap U'\neq\emptyset$, the coordinate change from one chart to the other on $U\cap U'$ for tensors is the following:*

$$F=F^{i_1,...,i_r}_{j_1,...,j_s}\frac{\partial}{\partial x^{i_1}}\otimes...\otimes\frac{\partial}{\partial x^{i_r}}\otimes\mathrm{d}x^{j_1}\otimes...\otimes\mathrm{d}x^{j_s}=\bar{F}^{i_1,...,i_r}_{j_1,...,j_s}\frac{\partial}{\partial \bar{x}^{i_1}}\otimes...\otimes\frac{\partial}{\partial \bar{x}^{i_r}}\otimes\mathrm{d}\bar{x}^{j_1}\otimes...\otimes\mathrm{d}\bar{x}^{j_s} \tag{B.1.11}$$

  with

$$\bar{F}^{i_1,...,i_r}_{j_1,...,j_s}=F^{l_1,...,l_r}_{k_1,...,k_s}\frac{\partial \bar{x}^{i_1}}{\partial x^{l_1}}\cdots\frac{\partial \bar{x}^{i_r}}{\partial x^{l_r}}\frac{\partial x^{k_1}}{\partial \bar{x}^{j_1}}\cdots\frac{\partial x^{k_s}}{\partial \bar{x}^{j_s}} \tag{B.1.12}$$

B.2. Basic Concepts on (pseudo-)Riemannian Geometry

Once the fundamentals of differential manifolds have been introduced, we will present the needed results and concepts on (pseudo-)Riemaniannian Geometry.

Definition 4. *Let M be a differential manifold. A* Riemannian metric *is a (0,2) tensor $g\in\mathfrak{T}^0_2(M)$ such that it is symmetric ($g(u,v)=g(v,u)$) and positive-definite (this is, for every $v\neq 0$, $g(v,v)>0$). If g is symmetric but not positive-definite,it will be called* pseudo-Riemannian metric. *(M,g) is said to be a* Riemannian manifold *(resp.* pseudo-Riemannian manifold*).*

It can be shown that every differential manifold with good enough topological characteristics[2] can be endowed with a Riemannian manifold.
Restricted to a single tangent space T_pM, g_p behaves as an scalar product, allowing us to calculate lengths, angles, volumes, etc. Given a set of local coordinates $(x^1, ..., x^n)$ the line element can be defined as

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{B.2.1}$$

And the volume of a region $A \subseteq M$ can be computed as

$$\mathrm{Vol}(A) = \int_A \sqrt{|\det g|}\mathrm{d}x^1...\mathrm{d}x^n, \tag{B.2.2}$$

where $\det g$ corresponds to the determinant of the matrix (g_{ij}).
The metric tensor further allows us to "transform" a (r, s) tensor into a different (r', s') type one, such that $r + s = r' + s'$:

Proposition B.2.1. *Given a (pseudo-)Riemannian manifold, there exists the following bijections:*

$$\begin{array}{rccc} \sharp: & \mathfrak{T}^r_s(M) & \to & \mathfrak{T}^{r+1}_{s-1}(M) \\ & f^{i_1,\dots,i_r}_{j_1,\dots,j_s} & \mapsto & \bar{f}^{i_1,\dots,i_r,\alpha}_{j_1,\dots,j_{s-1}} = f^{i_1,\dots,i_r}_{j_1,\dots,j_s} g^{j_s\alpha} \end{array} \qquad \begin{array}{rccc} \flat: & \mathfrak{T}^r_s(M) & \to & \mathfrak{T}^{r-1}_{s+1}(M) \\ & f^{i_1,\dots,i_r}_{j_1,\dots,j_s} & \mapsto & \bar{f}^{i_1,\dots,i_{r-1}}_{j_1,\dots,j_s,\alpha} = f^{i_1,\dots,i_r}_{j_1,\dots,j_s} g_{i_r\alpha} \end{array}, \tag{B.2.3}$$

where again Einstein's summation convention has been used, and $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}$. These operations are respectively called raising and lowering of indexes*.*

Additionally, it is possible to "contract" the number of indexes of a tensor:

Proposition B.2.2. *The following operation is well-defined:*

$$\begin{array}{ccc} \mathfrak{T}^r_s(M) & \to & \mathfrak{T}^{r-1}_{s-1}(M) \\ f^{i_1,\dots,i_r}_{j_1,\dots,j_s} & \mapsto & \bar{f}^{i_1,\dots,i_{r-1}}_{j_1,\dots,j_{s-1}} = f^{i_1,\dots,i_{r-1},\alpha}_{j_1,\dots,j_{s-1},\alpha} \end{array} \tag{B.2.4}$$

We can now generalize the concept of directional derivative from $\mathbb{R}^n$ to a general manifold. While in traditional calculus over $\mathbb{R}^n$ the derivative can be easily understood as the infinitesimal difference of a function over close points along a specific direction, this does not translate well to generic manifolds, as coordinate curves might not remain parallel. We introduce the following concept:

Definition 5. *Let M be a (not necessarily (pseudo-)Riemannian) differential manifold. A* covariant derivative *is a map*

$$\begin{array}{rccc} \nabla: & \mathfrak{X}(M) \times \mathfrak{X}(M) & \to & \mathfrak{X}(M) \\ & (X, Y) & \mapsto & \nabla_X Y \end{array} \tag{B.2.5}$$

verifying

- $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z, \quad \forall X, Y, Z \in \mathfrak{X}(M)$
- $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z, \quad \forall X, Y, Z \in \mathfrak{X}(M)$
- $\nabla_{fX} = f\nabla_X Y, \quad \forall X, Y \in \mathfrak{X}(M), f \in \mathfrak{F}(M)$
- $\nabla_X(fY) = X(f)Y + f\nabla_X Y, \quad \forall X, Y \in \mathfrak{X}(M), f \in \mathfrak{F}(M)$. *Here $X(f)$ is understood as the vector field $X = X^i\frac{\partial}{\partial x^i}$ acting as a differential operator, $X(f) = X^i\frac{\partial f}{\partial x^i}$.*

A covariant derivative is determined its connection coefficients *or* Christoffel symbols Γ^k_{ij}*:*

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ji}\frac{\partial}{\partial x^j} \tag{B.2.6}$$

Given a general tensor $T \in \mathfrak{T}^r_s(M)$ given by its coefficients $T^{i_1,\dots,i_r}_{j_1,\dots,j_s}$, its covariant derivative with respect to $\frac{\partial}{\partial x^\alpha}$ is given by

$$\nabla_{\frac{\partial}{\partial x^\alpha}} T^{i_1,\dots,i_r}_{j_1,\dots,j_s} = \frac{\partial}{\partial x^\alpha}T^{i_1,\dots,i_r}_{j_1,\dots,j_s} + \Gamma^{i_1}_{\beta\alpha}T^{\beta,\dots,i_r}_{j_1,\dots,j_s} + ... + \Gamma^{i_r}_{\beta\alpha}T^{i_1,\dots,\beta}_{j_1,\dots,j_s} - \Gamma^{\beta}_{j_1\alpha}T^{i_1,\dots,i_r}_{\beta,\dots,j_s} - \Gamma^{\beta}_{j_s\alpha}T^{i_1,\dots,i_r}_{j_1,\dots,\beta} \tag{B.2.7}$$

For scalar functions $f \in \mathfrak{F}(M)$ it is immediate that $\nabla_{\frac{\partial}{\partial x^\alpha}} f = \frac{\partial f}{\partial x^\alpha}$.

This definition of covariant derivative is too general to be useful, so usually additional properties are required. The most common covariant derivative used in Riemannian Geometry is the so called *Levi-Civita connection*:

[2]Namely being *Hausdorff* (every par of distinct points p, q in M admits two open sets $U, V \subseteq M$ such that $U \cap V = \emptyset$) and verifying the *Second Numerability Axiom* (M admits a countable atlas). Every manifold encountered in General Relativity has this properties.

Definition 6. *Given a (pseudo-)Riemannian manifold (M, g), the* ***unique*** *covariant derivative that is* symmetric *(also known as* torsion free, $\Gamma^k_{ij} = \Gamma^k_{ji}$*) and* metric *(this is, for every $X, Y, Z \in \mathfrak{X}(M)$ we have $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$) is called* Levi-Civita connection*, and its Christoffel symbols are given by*

$$\Gamma^k_{ij} = \Gamma^k_{ji} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} + \frac{\partial g_{kj}}{\partial x^i} \right) \tag{B.2.8}$$

We will denote covariant derivatives with respect with the coordinate vector fields $\frac{\partial}{\partial x^i}$ with a semicolon, while usual derivatives will be denoted with a normal comma:

$$\nabla_{\frac{\partial}{\partial x^\alpha}} T^{i_1,\dots,i_r}_{j_1,\dots,j_s} = T^{i_1,\dots,i_r}_{j_1,\dots,j_s;\alpha}, \qquad \frac{\partial}{\partial x^\alpha} T^{i_1,\dots,i_r}_{j_1,\dots,j_s} = T^{i_1,\dots,i_r}_{j_1,\dots,j_s,\alpha} \tag{B.2.9}$$

Once the fundamental concepts on which Riemannian Geometry is constructed, we shall give a grief overlook on its main subject: *curvature*. While the curvature of plane curves can be understood in terms of the radius of the tangent circles at each point, or in surfaces of the radii of the inscriptable circles. For general manifolds this generalization is much more complex, and requires the introduction of new tensors:

Definition 7. *Let (M, ∇) be a (pseudo-)Riemannian manifold endowed with the Levi-Civita connection. The* Riemann curvature tensor *R_{ijk} is defined as*

$$R^l_{ijk} = \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^m_{ik} \Gamma^l_{jm} + \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} \tag{B.2.10}$$

While it is not easy to find a direct interpretation of the information in R^l_{ijk}, it is possible to show that it affected only by intrinsic geometry*, that is, those geometric features which can be obtained from measuring distances, angles, etc., from "inside" the manifold, and is not affected by the way the manifold might be embedded in other higher-dimensional manifolds*[3].

The Riemann curvature tensor can be used to obtain new tensors which are useful in General Relativity:

Definition 8. *Let again (M, ∇) be a (pseudo-)Riemannian manifold endowed with the Levi-Civita connection. The* Ricci curvature tensor *R_{ik} is defined by the following contraction of the Riemann curvature tensor:*

$$R_{ik} = R^j_{ijk} \tag{B.2.11}$$

The Ricci scalar *is in turn defined as*

$$R = R^i_i = R_{ij} g^{ij} \tag{B.2.12}$$

The simplest geometric interpretation of the Ricci scalar is the amount the n-volume of a ball $\{\|x - p_0\| < \epsilon\}$ deviates from an analogous ball in $\mathbb{R}^n$. Manifolds with $R > 0$, such as spheres, will (locally) feature balls with greater volume than in the flat case, while the opposite occurs for $K < 0$ regions.

While these three tensors are all represented by the letter R, as the three are obtained from the Riemann curvature tensor, and are of different (r, s) type, they can be easily be differentiated by the number of indexes present.

Finally, we introduce the so-called *Einstein Tensor*, which is of special interest in General Relativity, acting as the left side of Einstein Equations:

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \tag{B.2.13}$$

B.3. Perturbed metric tensors and calculations thereof

In the final section of this Appendix we will obtain the perturbed Christoffel symbols $\delta\Gamma^\gamma_{\alpha\beta}$ associated to the metric perturbations $\delta g_{\mu\nu}$, which in turn will be used in the derivation of the perturbed Einstein Equations $\delta G^\mu_\nu = \delta T^\mu_\nu$ and conservation equations $T^\mu_{\nu;\mu} = 0$ in Chapter 3. While the appendix from [**Bardeen, 1980**] features the final expressions of the perturbed Ricci tensor δR^μ_ν, as well as the conservation equations, we will perform the intermediate calculations, both as a double-check and as a way to dive deeper into the world of metric perturbations. The theoretical basis behind this section can be found in Chapter 4 from [**Piattella, 2018**].

[3] An easy visualization of these concepts is the following. While a cylinder and a plane both are differently embedded in $\mathbb{R}^n$, the cylinder can be developed into a plane, and thus both have the same intrinsic curvature. On the contrary, a sphere cannot be developed into a plane, and thus its intrinsic geometry is different from that of the plane.

We will recall that, as introduced in Chapter 3, we introduced the following linear perturbations to the metric:

$$g_{\mu\nu} = \underbrace{a^2 \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & h_{ij} \end{array}\right]}_{\bar{g}_{\mu\nu}} + \underbrace{a^2 \left[\begin{array}{c|c} -2AU & -BV_i \\ \hline -BV_i^T & 2H_L U h_{ij} + 2H_T W_{ij} \end{array}\right]}_{\delta g_{\mu\nu}}, \tag{B.3.1}$$

where $\{A, B, H_L, H_T\}$ were functions depending only on time, h_{ij} is the metric of a 3-dimensional space with constant curvature K and U, V and W were the scalar, vector and tensor spatial harmonics obtained from the Hemholtz Equation $U_{;i}^{;i} + k^2 U = 0$. While the three $g_{\mu\nu}$, $\bar{g}_{\mu\nu}$ and $\delta g_{\mu\nu}$ correspond to 4×4 matrices, only g and $\bar{g}$ are pseudo-Riemannian metrics.
We will consider our perturbations to be small enough so that only linear terms on $\delta g_{\mu\nu}$ are considered. It is not guaranteed that $\delta g_{\mu\nu}$ is a non-singular matrix ($\det \delta g$ might be zero), so we will work with a "pseudo-inverse"

$$\delta g^{\mu\nu} := -\bar{g}^{\mu\rho} \delta g_{\rho\sigma} \bar{g}^{\sigma\nu} \tag{B.3.2}$$

Which is valid in our linear approach, as

$$\begin{aligned} g^{\mu\nu} g_{\nu\theta} &= (\bar{g}^{\mu\nu} + \delta g^{\mu\nu})(\bar{g}_{\nu\theta} + \delta g_{\nu\theta}) = (\bar{g}^{\mu\nu} - \bar{g}^{\mu\rho}\delta g_{\rho\sigma}\bar{g}^{\sigma\nu})(\bar{g}_{\nu\theta} + \delta g_{\nu\theta}) \\ &= \bar{g}^{\mu\nu}\bar{g}_{\nu\theta} + \bar{g}^{\mu\nu}\delta g_{\nu\theta} - \bar{g}^{\mu\rho}\delta g_{\rho\sigma} \underbrace{\bar{g}^{\sigma\nu}\bar{g}_{\nu\theta}}_{\delta^\sigma_\theta} - \underbrace{\bar{g}^{\mu\rho}\delta g_{\rho\sigma}\bar{g}^{\sigma\nu}\delta g_{\nu\theta}}_{\mathcal{O}(\delta g^2)} \\ &= \delta^\mu_\theta + \mathcal{O}(\delta g^2) \end{aligned} \tag{B.3.3}$$

The Christoffel symbols, fundamental in the calculus of the different curvature tensors, can be decomposed as

$$\begin{aligned} \Gamma^\mu_{\nu\rho} &= \frac{1}{2} g^{\mu\sigma}(g_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) \\ &= \frac{1}{2}\bar{g}^{\mu\sigma}(\bar{g}_{\sigma\nu,\rho} + \bar{g}_{\sigma\rho,\nu} - \bar{g}_{\nu\rho,\sigma}) + \frac{1}{2}\bar{g}^{\mu\sigma}(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma}) \\ &+ \frac{1}{2}\delta g^{\mu\sigma}(\bar{g}_{\sigma\nu,\rho} + \bar{g}_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) + \underbrace{\frac{1}{2}\delta g^{\mu\sigma}(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma})}_{\mathcal{O}(\delta g^2)} \end{aligned} \tag{B.3.4}$$

This way we can separate the Christoffel symbols as $\Gamma^\mu_{\nu\rho} = \bar{\Gamma}^\mu_{\nu\rho} + \delta\Gamma^\mu_{\nu\rho}$, where $\bar{\Gamma}^\mu_{\nu\rho}$ are the Christoffel symbols associated to the background metric $\bar{g}_{\mu\nu}$ and $\delta\Gamma^\mu_{\nu\rho}$ are the *perturbed Christoffel symbols*, immediately given by

$$\delta\Gamma^\mu_{\nu\rho} = \frac{1}{2}\bar{g}^{\mu\sigma}(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma} - 2\delta g_{\sigma\alpha}\bar{\Gamma}^\alpha_{\nu\rho}) \tag{B.3.5}$$

Regarding the background Christoffel symbols, it is an standard exercise in any Astrophysics/Cosmology course to show that the only non null $\bar{\Gamma}^\mu_{\nu\rho}$ in a FLRW background metric are

$$\bar{\Gamma}^0_{00} = \mathcal{H}, \quad \bar{\Gamma}^0_{ij} = \mathcal{H}\delta_{ij}, \quad \bar{\Gamma}^i_{0j} = \mathcal{H}\delta^i_j, \qquad \text{for} \quad i,j \in \{1,2,3\} \tag{B.3.6}$$

After a few boring but straightforward calculations, we obtain the perturbed Christoffel symbols:

$$\delta\Gamma^0_{00} = \dot{A}U \tag{B.3.7a}$$

$$\delta\Gamma^0_{i0} = [-kA - \mathcal{H}B]V_i \tag{B.3.7b}$$

$$\delta\Gamma^i_{00} = [-(\dot{B} + \mathcal{H}B) - kA]V_i \tag{B.3.7c}$$

$$\delta\Gamma^0_{ij} = kB\left(\frac{1}{3}h_{ij} - W_{ij}\right) + (\dot{H}_L U h_{ij} + \dot{H}_T W_{ij}) + \mathcal{H}(2H_L U h_{ij} + 2H_T W_{ij}) - 2\mathcal{H}\delta_{ij}AU \tag{B.3.7d}$$

$$\delta\Gamma^i_{j0} = (\dot{H}_L U \delta^i_j + \dot{H}_T W^i_j) \tag{B.3.7e}$$

$$\delta\Gamma^i_{jk} = \left[-kH_L(V_k\delta^i_j + V_j\delta^i_k - V^i\delta_{jk}) + H_T(W^i_{j,k} + W^i_{k,j} - W^{\ i}_{j,k})\right] + 2\mathcal{H}\delta_{jk}BV^i \tag{B.3.7f}$$

Now that all the Christoffel symbols have been obtained, we can work with the Ricci tensor. By contracting the Riemann curvature tensor (see eq. B.2.10) as $R_{\mu\nu} = R^\theta_{\mu\theta\nu}$, we can directly express the Ricci tensor in terms of the Christoffel symbols and their derivatives:

$$R_{\mu\nu} = \Gamma^\theta_{\mu\nu,\theta} - \Gamma^\theta_{\mu\theta,\nu} + \Gamma^\theta_{\mu\nu}\Gamma^\rho_{\theta\rho} - \Gamma^\theta_{\mu\rho}\Gamma^\rho_{\nu\theta} \tag{B.3.8}$$

As the Christoffel symbols are decomposed as $\Gamma^\mu_{\nu\sigma} = \bar{\Gamma}^\mu_{\nu\sigma} + \delta\Gamma^\mu_{\nu\sigma}$, we have

$$\begin{aligned} R_{\mu\nu} &= \bar{\Gamma}^\theta_{\mu\nu,\theta} + \delta\Gamma^\theta_{\mu\nu,\theta} - \bar{\Gamma}^\theta_{\mu\theta,\nu} - \delta\Gamma^\theta_{\mu\theta,\nu} + \bar{\Gamma}^\theta_{\mu\nu}\bar{\Gamma}^\rho_{\theta\rho} + \bar{\Gamma}^\theta_{\mu\nu}\delta\Gamma^\rho_{\theta\rho} + \delta\Gamma^\theta_{\mu\nu}\bar{\Gamma}^\rho_{\theta\rho} \\ &+ \underbrace{\delta\Gamma^\theta_{\mu\nu}\delta\Gamma^\rho_{\theta\rho}}_{\mathcal{O}(\delta g^2)} - \bar{\Gamma}^\theta_{\mu\rho}\bar{\Gamma}^\rho_{\nu\theta} - \bar{\Gamma}^\theta_{\mu\rho}\delta\Gamma^\rho_{\nu\theta} - \delta\Gamma^\theta_{\mu\rho}\bar{\Gamma}^\rho_{\nu\theta} - \underbrace{\delta\Gamma^\theta_{\mu\rho}\delta\Gamma^\rho_{\nu\theta}}_{\mathcal{O}(\delta g^2)} \end{aligned} \tag{B.3.9}$$

Taking into account only first order terms and using $\bar{R}_{\mu\nu} = \bar{\Gamma}^{\theta}_{\mu\nu,\theta} - \bar{\Gamma}^{\theta}_{\mu\theta,\nu} + \bar{\Gamma}^{\theta}_{\mu\nu}\bar{\Gamma}^{\rho}_{\theta\rho} - \bar{\Gamma}^{\theta}_{\mu\rho}\bar{\Gamma}^{\rho}_{\nu\theta}$ as the "background Ricci tensor", we define the *perturbed Ricci tensor* as

$$\delta R_{\mu\nu} := \delta\Gamma^{\theta}_{\mu\nu,\theta} - \delta\Gamma^{\theta}_{\mu\theta,\nu} + \bar{\Gamma}^{\theta}_{\mu\nu}\delta\Gamma^{\rho}_{\theta\rho} + \delta\Gamma^{\theta}_{\mu\nu}\bar{\Gamma}^{\rho}_{\theta\rho} - \bar{\Gamma}^{\theta}_{\mu\rho}\delta\Gamma^{\rho}_{\nu\theta} - \delta\Gamma^{\theta}_{\mu\rho}\bar{\Gamma}^{\rho}_{\nu\theta} \tag{B.3.10}$$

After another round of calculations, the perturbed Ricci tensor is found to have the following components

$$\delta R_{00} = \left[\frac{1}{2}Ak^2U + 3\mathcal{H}\dot{A}U - k(\dot{B} + \mathcal{H}B)U - 3(\ddot{H}_L + \mathcal{H}\dot{H}_L)\right]U \tag{B.3.11a}$$

$$\delta R_{0i} = \left\{-2k\mathcal{H}A + \frac{1}{2}B - \left[k^2 - \frac{2}{3}k\left(1 - \frac{3K}{k^2}\right)\right] - (\dot{\mathcal{H}} + \mathcal{H}^2)B + 2k\dot{H}_L V_i + \frac{2}{3}\dot{H}_T k\left(1 - \frac{3K}{k^2}\right)\right\} V_i \tag{B.3.11b}$$

$$\begin{aligned}\delta R_{ij} \quad = \quad & -AU_{,ij} - \mathcal{H}\dot{A}\delta_{ij} + (2\mathcal{H}^2 + \dot{\mathcal{H}})AU\delta_{ij} + k^2 H_L\left[\frac{2}{3}Uh_{ij} - W_{ij}\right] \\ & + k^2 H_T\left[W_{ij} + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\left(\frac{1}{3}h_{ij} - W_{ij}\right)\right] + (\ddot{H}_L U h_{ij} + \ddot{H}_T W_{ij}) + \tfrac{1}{2}2\mathcal{H}(\dot{H}_L U h_{ij} + \dot{H}_T W_{ij}) \\ & + (2\mathcal{H}^2 + \dot{\mathcal{H}})(H_L U h_{ij} + H_T W_{ij}) + \mathcal{H}(3\dot{H}_L + kB)U\delta_{ij} + k\left(\dot{B} - 2\mathcal{H}B\right)\left(\frac{1}{3}h_{ij}U - W_{ij}\right)\end{aligned} \tag{B.3.11c}$$

As exhausting as these expressions might look, we are not finished yet. Once the Ricci tensor is obtained, we can calculate the Ricci Scalar, which is obtained contracting the indexes of the Ricci Tensor:

$$R = g^{\mu\nu}R_{\mu\nu} = (\bar{g}^{\mu\nu} + \delta g^{\mu\nu})(\bar{R}_{\mu\nu} + \delta R_{\mu\nu}) = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} + \bar{g}^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu}\bar{R}_{\mu\nu} + \underbrace{\delta g^{\mu\nu}\delta R_{\mu\nu}}_{\mathcal{O}(\delta g^2)} \tag{B.3.12}$$

Denoting the background Ricci scalar $\bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}$, to first order we can split the total Ricci scalar as $R = \bar{R} + \delta R$, where the *perturbed Ricci scalar* has the following expression:

$$\delta R = \bar{g}^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu}\bar{R}_{\mu\nu} \tag{B.3.13}$$

Using that the only non-banishing components of the Ricci tensor are the following,

$$\bar{R}_{00} = -3\dot{\mathcal{H}}, \qquad \bar{R}_{ij} = \delta_{ij}(\dot{\mathcal{H}} + 2\mathcal{H}^2) \tag{B.3.14}$$

we can compute the perturbed Ricci scalar as

$$\begin{aligned}\delta R \quad = \quad & \frac{1}{a^2}\Big(2Ak^2 - 6\mathcal{H}\dot{A} - 12(\dot{\mathcal{H}} + \mathcal{H}^2)A + 2\dot{B}k + 6\mathcal{H}Bk \\ & + 6\ddot{H}_L U + 18\mathcal{H}\dot{H}_L + 2k^2H_L - 2k^2H_L + \frac{4}{3}H_T\left(k^2 - 3K\right)\Big)U\end{aligned} \tag{B.3.15}$$

Once the Ricci tensor and scalar have been obtained, we are in conditions to compute the Einstein tensor. We will do so in a mixed version version, which simplifies the expressions obtained when equated with the stress-energy tensor T^{μ}_{ν}

$$\begin{aligned}G^{\mu}_{\nu} \quad & = \quad g^{\mu\rho}G_{\rho\nu} = g^{\mu\rho}\left(R_{\rho\nu} - \frac{1}{2}g_{\rho\nu}R\right) = g^{\mu\nu}R_{\rho\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R = (\bar{g}^{\mu\rho} + \delta g^{\mu\rho})(\bar{R}_{\rho\nu} + \delta R_{\rho\nu}) - \frac{1}{2}\delta^{\mu}_{\nu}(\bar{R} + \delta R) \\ & = \quad \bar{g}^{\mu\rho}\bar{R}_{\rho\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\bar{R} + \bar{g}^{\mu\rho}\delta R_{\rho\nu} + \delta g^{\mu\rho}\bar{R}_{\rho\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\delta R + \underbrace{\delta g^{\mu\rho}\delta R_{\rho\nu}}_{\mathcal{O}(\delta g^2)}\end{aligned} \tag{B.3.16}$$

Using $\bar{G}^{\mu}_{\nu} = barg^{\mu\rho}\bar{R}_{\rho\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\bar{R}$ as the "background Einstein tensor", we have the following expression for the perturbed one:

$$\delta G^{\mu}_{\nu} = \bar{g}^{\mu\rho}\delta R_{\rho\nu} + \delta g^{\mu\rho}\bar{R}_{\rho\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\delta R, \tag{B.3.17}$$

which has the following components:

$$\delta G^{0}_{0} = \frac{3}{a^2}\left\{-2\mathcal{H}\dot{H}_L + (\mathcal{H}^2 - \dot{\mathcal{H}})A - \frac{2k}{3}\mathcal{H}B - \frac{1}{3}(k^2 - 3K)\left(H_L + \frac{1}{3}H_T\right)\right\}U \tag{B.3.18a}$$

$$\delta G^{0}_{i} = \frac{2}{a^2}\left[-k\dot{H}_L - \frac{k}{3}\left(1 - \frac{3K}{k^2}\right)\dot{H}_T + k\mathcal{H}A - KB\right]V_i \tag{B.3.18b}$$

$$\begin{aligned}\delta G^{i}_{j} \quad = \quad & \frac{1}{a^2}\left\{\frac{1}{3}(k^2 - 3K)\left(H_L + \frac{1}{3}H_T\right) - 2\ddot{H}_L - \mathcal{H}(4\dot{H}_L - 2\dot{A}) + \left[\frac{-2k^2}{3} - (\mathcal{H}^2 - \dot{\mathcal{H}})\right]A + \frac{k}{3}\left(-2\dot{B} - 4\mathcal{H}B\right)\right\}\delta^{i}_{j}U \\ & + \frac{1}{a^2}\left[\ddot{H}_T + 2\mathcal{H}\dot{H}_T - k\left(\dot{B} + 2\mathcal{H}B\right) - k^2\left(H_L + \frac{1}{3}H_T + A\right)\right]W^{i}_{j}\end{aligned} \tag{B.3.18c}$$

We can finally move on to the conservation equations obtained from the *divergence-free* nature of the stress-energy tensor:

$$T^{\mu}_{\nu;\mu} = \frac{\partial}{\partial x^{\mu}} T^{\mu}_{\nu} + \Gamma^{\mu}_{\alpha\mu} T^{\alpha}_{\nu} - \Gamma^{\alpha}_{\nu\mu} T^{\mu}_{\alpha} = 0 \tag{B.3.19}$$

As we did in the previous calculations, we will only be dealing with first order terms, what means that we will consider terms of the kind $\delta\Gamma^{\alpha}_{\cdot\cdot}\delta T^{\cdot}_{\alpha}$ as second order.

We begin with the *energy equation*, $T^{\mu}_{0;\mu} = 0$:

$$\frac{\partial}{\partial \tau} T^0_0 + \frac{\partial}{\partial x^i} T^i_0 + \Gamma^{\alpha}_{0\alpha} T^0_0 + \underbrace{\Gamma^{\alpha}_{i\alpha} T^i_0}_{\mathcal{O}(\delta g \delta T)} - \Gamma^0_{00} T^0_0 - \underbrace{\Gamma^0_{i0} T^i_0}_{\mathcal{O}(\delta g \delta T)} - \underbrace{\Gamma^i_{00} T^0_i}_{\mathcal{O}(\delta g \delta T)} - \Gamma^i_{j0} T^j_i = 0 \tag{B.3.20}$$

This way,

$$\begin{aligned} 0 &= \frac{\partial}{\partial \tau}[\bar{\rho}(1+\delta U)] + (\bar{\rho}+\bar{P})(v-B)\underbrace{\frac{\partial}{\partial x^i} V^i}_{kU} + \left(\mathcal{H} + \dot{A}U + 3\mathcal{H} + 3\dot{H}_L U\right)[\bar{\rho}(1+\delta U)] \\ &- (\mathcal{H} + \dot{A}U)[\bar{\rho}(1+\delta U)] + (\mathcal{H} + \dot{H}_L U)\delta^i_j [\bar{P}(1+\pi_L U)\delta^j_i] \end{aligned} \tag{B.3.21}$$

We now consider only the linear perturbation terms (those with U) and consider $U^2 \sim 0$. Multiplying by a^3 and grouping terms, we arrive to the following equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar{\rho} a^3 \delta) + (\bar{\rho}+\bar{P}) a^3 (kv - kB + 3\dot{H}_L) + 3\bar{P} a^2 \dot{a} \pi_L = 0 \tag{B.3.22}$$

And we do the same with the *momentum equation*, $T^{\mu}_{i;\mu} = 0$:

$$\frac{\partial}{\partial \tau} T^0_i + \frac{\partial}{\partial x^j} T^j_i + \Gamma^{\mu}_{0\mu} T^0_i + \Gamma^{\mu}_{\mu j} T^j_i - \Gamma^0_{0i} T^0_0 - \Gamma^0_{ji} T^j_0 - \Gamma^j_{0i} T^0_j - \Gamma^j_{ki} T^k_j = 0 \tag{B.3.23}$$

In this case it is not immediate that any of the above terms can be deprecated. Working carefully and only taking into account first order elements,

$$\begin{aligned} 0 &= \Big[\underbrace{\frac{\partial}{\partial \tau}(\bar{\rho}+\bar{P})}_{-3\mathcal{H}(\bar{\rho}+\bar{P})(1+c_s^2)} (v-B) + (\bar{\rho}+\bar{P})\tfrac{\mathrm{d}}{\mathrm{d}\tau}(v-B)\Big] V_i + \bar{P}\left(\pi_L U_{,j}\delta^j_i + \pi_T W^j_{i,j}\right) + 4\mathcal{H}(\bar{\rho}+\bar{P})(v-B)V_i \\ &- AU_{,j}\delta^j_i(\bar{\rho}+\bar{P} \end{aligned} \tag{B.3.24}$$

Now, using that $U_{,i} = -kV_i$ and that $W^j_{i,j} = \frac{2}{3}k(1-\frac{3K}{k^2})V_i$ (immediate from eq. 3.1.4), we have

$$(\bar{\rho}+\bar{P})\left[\frac{\mathrm{d}}{\mathrm{d}\tau}(v-B) + \mathcal{H}(1-3c_s^2)(v-B) - kA\right] - k\bar{P}\pi_L + \frac{2}{3}k\left(1-\frac{3K}{k^2}\right)\bar{P}\pi_T = 0 \tag{B.3.25}$$

If $\omega \neq -1$, we would have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(v-B) + \mathcal{H}(1-3c_s^2)(v-B) - kA - k\frac{\omega}{\omega+1}\pi_L + \frac{2}{3}k\left(1-\frac{3K}{k^2}\right)\frac{\omega}{\omega+1}\pi_T = 0 \tag{B.3.26}$$

On the other hand, if $\omega = -1$ and $\bar{\rho} = -\bar{P}$, the expression simplifies greatly to

$$\pi_L = \frac{2}{3}\left(1-\frac{3K}{k^2}\right)\pi_T \tag{B.3.27}$$

APPENDIX C

Solving the evolution equations

Contents

In this appendix we will perform the calculations needed to solve eq. 4.1.9 for the different standard values of the equation of state ω. We will then obtain the power spectrum associated with density evolution. See Appendix A for the definition of the power spectrum of a random field and some properties of the Fourier Transform. Our evolution equation will take the form described by eq. 4.1.9:

$$a^2 \frac{\partial^2 \varepsilon}{\partial a^2} - \frac{3\omega}{C} a^{3\omega+1} \nabla^2 \varepsilon + \frac{3(\omega+1)}{2} \left(a \frac{\partial \varepsilon}{\partial a} - \varepsilon \right) = 0, \qquad \text{with} \quad C = \bar{\rho}_0 a_0^{3(\omega+1)} \tag{C.0.1}$$

In order to solve the equations, we will perform a Fourier transform on them, thus having to solve a ODE with a k parameter instead than a PDE where ∇^2 consists in second order spatial derivatives. Under the homogeneity and isotropy of the perturbation field described in the main text, we will consider the Fourier modes of the solutions to depend only on $k = |\vec{k}|$ instead than on $\vec{k}$.

C.1. Inflationary Era/Λ-dominated ($\omega = -1$)

We begin with a universe dominated by dark energy or in a inflationary process, with $\omega = -1$ as the equation of state. This way, eq. 4.1.9 looks the following way:

$$a^2 \frac{\partial^2 \varepsilon}{\partial a^2} + \frac{3}{C} a^{-2} \nabla^2 \varepsilon = 0, \qquad \text{with} \quad C = \bar{\rho}_* \tag{C.1.1}$$

Doing the following change of variable[1],

$$z := \sqrt{\frac{3}{C}} a^{-1} \tag{C.1.2}$$

we can express the density derivatives with respect to a as:

$$\frac{\partial \varepsilon}{\partial a} = \frac{\partial \varepsilon}{\partial z} \frac{\partial z}{\partial a} = -\sqrt{\frac{3}{C}} a^{-2} \frac{\partial \varepsilon}{\partial z} \tag{C.1.3a}$$

$$\frac{\partial^2 \varepsilon}{\partial a^2} = \frac{3}{C} a^{-4} \frac{\partial^2 \varepsilon}{\partial z^2} + 2\sqrt{\frac{3}{C}} a^{-3} \frac{\partial \varepsilon}{\partial z} \tag{C.1.3b}$$

Multiplying this last equation by a^4, we can express $\frac{\partial^2 \varepsilon}{\partial a^2}$ in terms of z:

$$a^4 \frac{\partial^2 \varepsilon}{\partial a^2} = \frac{3}{C} \frac{\partial^2 \varepsilon}{\partial z^2} + 2\sqrt{\frac{3}{C}} a \frac{\partial \varepsilon}{\partial z} = \frac{3}{C} \left(\frac{\partial^2 \varepsilon}{\partial z^2} + \frac{2}{z} \frac{\partial \varepsilon}{\partial z} \right), \tag{C.1.4}$$

using the definition of z in the last equality. Now, if we multiply eq. C.1.1 by a^2, then it is easy to see that

$$0 = a^4 \frac{\partial^2 \varepsilon}{\partial a^2} + \frac{3}{C} \nabla^2 \varepsilon = \frac{3}{C} \left(\frac{\partial^2 \varepsilon}{\partial z^2} + \frac{2}{z} \frac{\partial \varepsilon}{\partial z} \right) + \frac{3}{C} \nabla^2 \varepsilon \tag{C.1.5}$$

This is,

$$\frac{\partial^2 \varepsilon}{\partial z^2} + \frac{2}{z} \frac{\partial \varepsilon}{\partial z} + \nabla^2 \varepsilon = 0 \tag{C.1.6}$$

We now introduce another variable change:

$$y := z\varepsilon = \sqrt{\frac{3}{C}} \frac{\varepsilon}{a} \tag{C.1.7}$$

With the following derivatives with respect to z:

$$\frac{\partial y}{\partial z} = \varepsilon + z \frac{\partial \varepsilon}{\partial z} \tag{C.1.8a}$$

[1] The variable z has nothing to do with redshift, but is rather used as an intermediate step in our way towards the solution.

$$\frac{\partial^2 y}{\partial z} = 2\frac{\partial \varepsilon}{\partial z} + z\frac{\partial^2 \varepsilon}{\partial z^2} = z\left(\frac{\partial^2 \varepsilon}{\partial z^2} + \frac{2}{z}\frac{\partial \varepsilon}{\partial z}\right) = -z\nabla^2\varepsilon = -\nabla^2 y \tag{C.1.8b}$$

where we have used eq. C.1.6 and the fact that, as z is only function of a, which is spatially constant, $\nabla^2 z = 0$. This way, y will be the solution of the following equation:

$$\frac{\partial^2}{\partial z^2} y + \nabla^2 y = 0 \tag{C.1.9}$$

We now, using our Fourier sign convention, expand $y(z)$ from eq. C.1.9 in Fourier modes $\hat{y}(z, k)$. From the Fourier transform of the equation,

$$\frac{\partial^2}{\partial^2 z}\hat{y}(z, k) - k^2\hat{y}(z, k) = 0 \tag{C.1.10}$$

we can express $\hat{y}(z, k)$ as

$$\hat{y}(z, k) = A(k)\cosh(k(z - z_*)) + B(k)\sinh(k(z - z_*)) \tag{C.1.11}$$

where $A(k)$ and $B(k)$ are constants depending on k and $k = |k|$. As we shall see soon, the term $z - z_*$, which does not alter the expression as it could have been grouped with $A(k)$ and $B(k)$, will simplify the expression of $\hat{y}(z, k)$ in terms of the initial conditions. To do this, we define the following intermediate functions:

$$\hat{\varphi}(z, k) := \hat{y}(z, k) = A(k)\cosh(k(z - z_*)) + B(k)\sinh(k(z - z_*)) \tag{C.1.12a}$$

$$\hat{\phi}(z, k) := \frac{1}{k}\frac{\partial}{\partial z}\hat{y}(z, k) = A(k)\sinh(k(z - z_*)) + B(k)\cosh(k(z - z_*)) \tag{C.1.12b}$$

Their initial values, at $z = z_*$, can be easily shown to be

$$\hat{\varphi}_*(k) := \hat{\varphi}(z_*, k) = A(k) \tag{C.1.13a}$$

$$\hat{\phi}_*(k) := \hat{\phi}(z_*, k) = B(k) \tag{C.1.13b}$$

so, the evolution of the $\hat{\varphi}$ and $\hat{\phi}$ terms is governed by the following simple equation:

$$\begin{pmatrix} \hat{\varphi}(z, k) \\ \hat{\phi}(z, k) \end{pmatrix} = \underbrace{\begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}}_{L(z,z_*)} \begin{pmatrix} \hat{\varphi}_*(k) \\ \hat{\phi}_*(k) \end{pmatrix}, \qquad \text{with } \alpha := k(z - z_*) \tag{C.1.14}$$

From eq. C.1.2, we have that $z = \sqrt{\frac{3}{\Lambda}}a^{-1}$, so z and a have a inversely proportional behavior. Moreover, as in natural units $\Lambda = 3H^2\Omega_\Lambda$, with $H = H_*$, $\Omega_\Lambda = 1$ for the inflationary period, and $H = H_0$, $\Omega_\Lambda = 0.6889 \pm 0.0056$ for the Λ-dominated era, we have

$$\alpha = \frac{k}{\sqrt{\Omega_\Lambda}}\frac{1}{H}\left(\frac{1}{a} - \frac{1}{a_*}\right) \tag{C.1.15}$$

We could be tempted to approximate these hyperbolic functions simply by their growing exponential terms, assuming a large value for $\alpha \propto -k(a^{-1} - a_*^{-1})$. This however might not always be the case, as for example k might take very small values, or in the case of the Λ-dominated era, where the scale factor a ranges from $a_m = 0.767$ and $a_0 = 1$, where the decaying exponential contributions might be relevant. Taking into account that, as $a \geq a_*$, $\alpha \leq 0$, the $L(z, z_*)$ matrix can be expressed as

$$L(z, z_*) = \begin{bmatrix} \cosh|\alpha| & -\sinh|\alpha| \\ -\sinh|\alpha| & \cosh|\alpha| \end{bmatrix} \tag{C.1.16}$$

We are now in the right conditions to study the power spectrum of these perturbations. To obtain δ_M, we just use the equation 4.1.2:

$$\delta_M = \frac{\varepsilon}{\bar{\rho}_{eff}(a)a^3} = C_{eff}^{-1}\varepsilon a^{3\omega_{eff}}, \qquad \text{whith} \quad C_{eff} = \bar{\rho}_{eff,0}a_0^{3(1+\omega_{eff})} \tag{C.1.17}$$

where $\bar{\rho}_{eff}$ corresponds to the density of the dominant fluid susceptible to suffer perturbations, being ω_{eff} its equation of state. We will obtain the perturbation evolution during a $\omega = -1$ era for different ω_{eff} options.

C.1.1. Effective Equation of State $\omega_{eff} = -1$. If we consider $\omega = \omega_{eff} = -1$, then the gauge invariant density perturbations take the following form:

$$\delta_M = \frac{\varepsilon}{C_{eff}a^3} = \sqrt{\frac{C}{27}}z^2 y, \tag{C.1.18}$$

where $C_{eff} = C = \Lambda$, the value of the cosmological constant. Performing the Fourier transform on δ_M, we have that

$$\hat{\delta}_M(z,k) = \sqrt{\frac{C}{27}}z^2\hat{y}(z,k) = \sqrt{\frac{C}{27}}z^2\hat{\varphi}(z,k) \tag{C.1.19}$$

Recall that z depends only on a and thus it is left invariant by the Fourier transform. We can then define a power term $\hat{\mu}_M$ (which is not the $\hat{\gamma}_M$ we will be working with in the final expression):

$$\hat{\mu}_M(z,k) := \frac{1}{k}\frac{\partial}{\partial z}\hat{\delta}_M(z,k) = \sqrt{\frac{C}{27}}z^2\left[\frac{2}{kz}\hat{\varphi}(z,k) + \hat{\phi}(z,k)\right] \tag{C.1.20}$$

By using eq. C.1.14, we have that

$$\hat{\delta}_M(z,k) = \sqrt{\frac{C}{27}}z^2\left[\cosh|\alpha|\hat{\varphi}_*(k) - \sinh|\alpha|\hat{\phi}_*(k)\right] \tag{C.1.21a}$$

$$\hat{\mu}_M(z,k) = \sqrt{\frac{C}{27}}z^2\left[\left(\frac{2}{kz}\hat{\varphi}_*(k) + \hat{\phi}_*(k)\right)\cosh|\alpha| - \left(\frac{2}{kz}\hat{\phi}_*(k) + \hat{\varphi}_*(k)\right)\sinh|\alpha|\right] \tag{C.1.21b}$$

As we are dealing we the evolution of these terms, we want to write $\hat{\delta}_M$ and $\hat{\mu}_M$ in terms of the initial density and "power" perturbations:

$$\begin{cases} \hat{\delta}_{M,p}(k) &= \hat{\delta}_M(z_*,k) &= \sqrt{\frac{C}{27}}z_*^2\hat{\varphi}_*(k) \\ \hat{\mu}_{M,p}(k) &= \hat{\mu}_M(z_*,k) &= \sqrt{\frac{C}{27}}z_*^2\left[\frac{2}{kz_*}\hat{\varphi}_*(k) + \hat{\phi}_*(k)\right] \end{cases} \tag{C.1.22}$$

from which we can find the expression of φ_* and ϕ_* in terms of $\hat{\delta}_{M,*}$ and $\hat{\mu}_{M,*}$:

$$\begin{cases} \hat{\varphi}_*(k) &= \sqrt{\frac{27}{C}}z_*^{-2}\hat{\delta}_{M,*}(k) \\ \hat{\phi}_*(k) &= \sqrt{\frac{27}{C}}z_*^{-2}\left(\hat{\mu}_{M,*}(k) - \frac{2}{kz_*}\hat{\delta}_{M,*}(k)\right) \end{cases} \tag{C.1.23}$$

After some calculations, we find that the density and power perturbations will evolve as

$$\hat{\delta}_M(z,k) = \left(\frac{z}{z_*}\right)^2\left[\left(\cosh|\alpha| + \frac{2}{kz_*}\sinh|\alpha|\right)\hat{\delta}_{M,*}(k) - \sinh|\alpha|\hat{\mu}_{M,*}(k)\right] \tag{C.1.24a}$$

$$\hat{\mu}_M(z,k) = \left(\frac{z}{z_*}\right)^2\left\{\left[\frac{2}{k}\left(\frac{1}{z} - \frac{1}{z_*}\right)\cosh|\alpha| - \left(1 - \frac{4}{k^2zz_*}\right)\sinh|\alpha|\right]\hat{\delta}_{M,*}(k) + \left[\cosh|\alpha| + \frac{2}{kz}\sinh|\alpha|\right]\hat{\mu}_{M,*}(k)\right\} \tag{C.1.24b}$$

We are finally able to compute the power spectrum, as defined in Appendix B, of these perturbations as

$$\Delta^2_{\delta_M}(z,k) = \left(\frac{z}{z_*}\right)^4\left[\left(\cosh|\alpha| + \frac{2}{kz_*}\sinh|\alpha|\right)^2\Delta^2_{\delta_M,*}(k) + \sinh^2|\alpha|\Delta^2_{\mu_M,*}(k) + \left(\sinh|2\alpha| + \frac{4}{kz_*}\sinh^2|\alpha|\right)\Delta^2_{\xi,p}(k)\right] \tag{C.1.25a}$$

$$\begin{aligned} \Delta^2_{\mu_M}(z,k) \quad = \quad & \left(\frac{z}{z_*}\right)^4\left\{\left[\frac{2}{k}\left(\frac{1}{z} - \frac{1}{z_*}\right)\cosh|\alpha| - \left(1 - \frac{4}{k^2zz_*}\right)\sinh|\alpha|\right]^2\Delta^2_{\delta_M,*}(k)\right. \\ & + \left[\cosh|\alpha| + \frac{2}{kz}\sinh|\alpha|\right]^2\Delta^2_{\mu_M,*}(k) \\ & \left. +2\left[\frac{2}{k}\left(\frac{1}{z} - \frac{1}{z_*}\right)\cosh|\alpha| - \left(1 - \frac{4}{k^2zz_*}\right)\sinh|\alpha|\right]\left[\cosh|\alpha| + \frac{2}{kz}\sinh|\alpha|\right]\Delta^2_{\xi,p}(k)\right\} \end{aligned} \tag{C.1.25b}$$

$$\begin{aligned} \Delta^2_{\xi}(z,k) \quad = \quad & \left(\frac{z}{z_*}\right)^4\left\{\left(\cosh|\alpha| + \frac{2}{kz_*}\sinh|\alpha|\right)\left[\frac{2}{k}\left(\frac{1}{z} - \frac{1}{z_*}\right)\cosh|\alpha| - \left(1 - \frac{4}{k^2zz_*}\right)\sinh|\alpha|\right]\Delta^2_{\delta_M,*}(k)\right. \\ & \left. - \left[\frac{1}{2}\sinh|2\alpha| + \frac{2}{kz}\sinh^2|\alpha|\right]\Delta^2_{\mu_M,*}(k) + \left[\cosh^2|\alpha| + \sinh^2|\alpha|\right]\Delta^2_{\xi,p}(k)\right\} \end{aligned} \tag{C.1.25c}$$

where $\Delta^2_{\delta_M,*}$, $\Delta^2_{\mu_M,*}$ and Ξ_* are the initial values, at a_* of this power spectrum coefficients.
We have to take into account that, when defining $\hat{\mu}_M$, and thus $\Delta^2_{\mu_M}$, we used $k^{-1}\frac{\partial}{\partial z}\hat{\delta}_M$, rather than $\hat{\gamma}_M = \frac{\partial}{\partial \ln a}\hat{\delta}_M$, see eq. C.1.20. This way,

$$\hat{\mu}_M = \frac{1}{k}\frac{\partial\hat{\delta}_M}{\partial z} = \frac{1}{k}\left(\frac{\partial z}{\partial a}\right)^{-1}\frac{\partial\hat{\delta}_M}{\partial a} = \frac{1}{k}\left(-\frac{z}{a}\right)^{-1}\frac{\partial\hat{\delta}_M}{\partial a} = -\frac{1}{kz}\frac{\partial\hat{\delta}_M}{\partial \ln a} = -\frac{\hat{\gamma}_M}{kz} \Leftrightarrow \hat{\gamma}_M = -kz\hat{\mu}_M \quad \text{(C.1.26)}$$

We can now "update" the power coefficients to their standard expression, which we will denote using $'$:

$$(\Delta^2_{\delta_M,*})'(k) = \Delta^2_{\delta_M,*}(k), \qquad (\Delta^2_{\gamma_M,*})'(k) = (kz_*)^2\Delta^2_{\mu_M,*}(k), \qquad (\Xi_*)'(k) = -kz_*\Xi_*(k) \quad \text{(C.1.27)}$$

Now,

$$\hat{\delta}_M(a,k) = \left(\frac{a_*}{a}\right)^2\left[\left(\cosh|\alpha| + \frac{2Ha_*}{k}\sinh|\alpha|\right)\hat{\delta}_{M,*}(k) + \frac{Ha_*}{k}\sinh|\alpha|\gamma_{M,*}(k)\right] \quad \text{(C.1.28a)}$$

$$\begin{aligned}\hat{\gamma}_M(a,k) \quad = \quad & -\frac{k}{Ha}\left(\frac{a_*}{a}\right)^2\left\{\left[\frac{2H}{k}(a-a_*)\cosh|\alpha| - \left(1-\frac{4H^2}{k^2}a_*a\right)\sinh|\alpha|\right]\hat{\delta}_{M,*}(k)\right.\\ & \left. -\frac{Ha_*}{k}\left[\cosh|\alpha| + \frac{2H_*a}{k}\sinh|\alpha|\right]\hat{\gamma}_{M,*}(k)\right\}\end{aligned} \quad \text{(C.1.28b)}$$

with the power coefficients being:

$$\begin{aligned}\Delta^2_{\delta_M}(a,k) \quad = \quad & \left(\frac{a_*}{a}\right)^4\left[\left(\cosh|\alpha| + \frac{2Ha_*}{k}\sinh|\alpha|\right)^2\Delta^2_{\delta_M,*}(k) + \frac{H^2a_*^2}{k^2}\sinh^2|\alpha|\Delta^2_{\gamma_M,*}(k)\right.\\ & \left. +\frac{Ha_*}{k}\left(\sinh|2\alpha| + \frac{4Ha_*}{k}\sinh^2|\alpha|\right)\Xi_*(k)\right]\end{aligned} \quad \text{(C.1.29a)}$$

$$\begin{aligned}\Delta^2_{\gamma_M}(a,k) \quad = \quad & \left(\frac{a_*}{a}\right)^4\frac{k^2}{H^2a^2}\left\{\left[\frac{2H}{k}(a-a_*)\cosh|\alpha| - \left(1-\frac{4H^2}{k^2}a_*a\right)\sinh|\alpha|\right]^2\Delta^2_{\delta_M,*}(k)\right.\\ & +\frac{H^2a_*^2}{k^2}\left[\cosh|\alpha| + \frac{2Ha_*}{k}\sinh|\alpha|\right]^2\Delta^2_{\gamma_M,*}(k)\\ & \left. -2\frac{Ha_*}{k}\left[\frac{2H}{k}(a-a_*)\cosh|\alpha| - \left(1-\frac{4H^2}{k^2}a_*a\right)\sinh|\alpha|\right]\left[\cosh|\alpha| + \frac{2H}{k}a\sinh|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \quad \text{(C.1.29b)}$$

$$\begin{aligned}\Xi(a,k) \quad = \quad & \left(\frac{a_*}{a}\right)^4\frac{k}{Ha}\left\{\left(\cosh|\alpha| + \frac{2Ha_*}{k}\sinh|\alpha|\right)\left[\frac{2H}{k}(a-a_*)\cosh|\alpha| - \left(1-\frac{4H^2}{k^2}a_pa\right)\sinh|\alpha|\right]\Delta^2_{\delta_M,*}(k)\right.\\ & \left. +\frac{H^2a_*^2}{k^2}\left[\frac{1}{2}\sinh|2\alpha| + \frac{2H}{k}a\sinh^2|\alpha|\right]\Delta^2_{\gamma_M,*}(k) - \frac{Ha_*}{k}\left[\cosh^2|\alpha| + \sinh^2|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \quad \text{(C.1.29c)}$$

C.1.2. Effective Equation of State $\omega_{eff} = 0$. We now consider a dust-like fluid as the substrate for the density perturbations. In this case, δ_M takes the following form:

$$\delta_M = \frac{\varepsilon}{C_{eff}} = C_{eff}^{-1}\frac{y}{z}, \qquad \text{with} \quad C_{eff} = \bar{\rho}_{eff,0}a_0^3 \quad \text{(C.1.30)}$$

Performing the Fourier transform on δ_M, it can be written as

$$\hat{\delta}_M(z,k) = C_{eff}\frac{\hat{y}(z,k)}{z} = C_{eff}^{-1}\frac{\hat{\varphi}(z,k)}{z} \quad \text{(C.1.31)}$$

Regarding the power term, $\hat{\mu}_M$,

$$\hat{\mu}_M(z,k) := \frac{1}{k}\frac{\partial}{\partial z}\hat{\delta}_M(z,k) = C_{eff}^{-1}z^{-1}\left[\hat{\phi}(z,k) - \frac{1}{kz}\hat{\varphi}(z,k)\right] \quad \text{(C.1.32)}$$

We now use eq. C.1.14,

$$\hat{\delta}_M(z,k) = C_{eff}^{-1}z^{-1}\left[\cosh|\alpha|\hat{\varphi}_*(k) - \sinh|\alpha|\hat{\phi}_*(k)\right] \quad \text{(C.1.33a)}$$

$$\hat{\mu}_M(z,k) = C_{eff}^{-1}z^{-1}\left[\left(\hat{\phi}_*(k) - \frac{1}{kz}\hat{\varphi}_*(zk)\right)\cosh|\alpha| + \left(\frac{1}{kz}\hat{\phi}_*(k) - \hat{\varphi}_*(k)\right)\sinh|\alpha|\right] \quad \text{(C.1.33b)}$$

We now obtain the initial values $\hat{\varphi}_*(k)$ and $\hat{\phi}(k)$ in terms of $\hat{\delta}_{M,*}(k)$ and $\hat{\mu}_{M,*}(k)$:

$$\left\{\begin{array}{rcl}\hat{\delta}_{M,*}(k) & = & C_{eff}^{-1}z_*^{-1}\hat{\varphi}_*(k)\\ \hat{\mu}_{M,*}(k) & = & C_{eff}^{-1}z_*^{-1}\left[\hat{\phi}_*(k) - \frac{1}{kz_*}\hat{\varphi}_*(k)\right]\end{array}\right. \Longrightarrow \left\{\begin{array}{rcl}\hat{\varphi}_*(k) & = & C_{eff}z_*\hat{\delta}_{M,*}(k)\\ \hat{\phi}_*(k) & = & C_{eff}z_*\hat{\mu}_{M,*}(k) + \frac{C_{eff}}{k}\hat{\delta}_{M,*}(k)\end{array}\right. \quad \text{(C.1.34)}$$

After some calculations, the expressions for the density and power perturbations are the following:

$$\hat{\delta}_M(z,k) = \frac{z_*}{z}\left[\left(\cosh|\alpha| - \frac{1}{kz_*}\sinh|\alpha|\right)\hat{\delta}_{M,*}(k) - \sinh|\alpha|\hat{\mu}_{M,*}(k)\right] \tag{C.1.35a}$$

$$\hat{\mu}_M(z,k) = \frac{z_*}{z}\left\{\left[\frac{1}{k}\left(z_*^{-1} - z^{-1}\right)\cosh|\alpha| + \left(\frac{1}{k^2 z_* z} - 1\right)\sinh|\alpha|\right]\hat{\delta}_{M,*}(k) + \left[\cosh|\alpha| + \frac{1}{kz}\sinh|\alpha|\right]\hat{\mu}_{M,*}(k)\right\} \tag{C.1.35b}$$

We can now compute the power spectrum associated to the perturbations.

$$\begin{aligned}\Delta^2_{\delta_M}(z,k) \quad = \quad & \left(\frac{z_*}{z}\right)^2\left[\left(\cosh|\alpha| - \frac{1}{kz_*}\sinh|\alpha|\right)^2\Delta^2_{\delta_M,*}(k) + \sinh^2|\alpha|\Delta^2_{\mu_M,*}(k)\right.\\ & \left.+2\left(d\frac{1}{kz_*}\sinh|\alpha| - \cosh|\alpha|\right)\sinh|\alpha|\Xi_*(k)\right]\end{aligned} \tag{C.1.36a}$$

$$\begin{aligned}\Delta^2_{\mu_M}(z,k) \quad = \quad & \left(\frac{z_*}{z}\right)^2\left\{\left[\tfrac{1}{k}\left(z_*^{-1} - z^{-1}\right)\cosh|\alpha| + \left(\tfrac{1}{k^2 z_* z} - 1\right)\sinh|\alpha|\right]^2\Delta^2_{\delta_M,*}(k) + \left[\cosh|\alpha| + \tfrac{1}{kz}\sinh|\alpha|\right]^2\Delta^2_{\mu_M,*}(k)\right.\\ & \left.+2\left[\tfrac{1}{k}\left(z_*^{-1} - z^{-1}\right) + \left(\tfrac{1}{k^2 z_* z} - 1\right)\sinh|\alpha|\right]\left[\cosh|\alpha| + \tfrac{1}{kz}\sinh|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{C.1.36b}$$

$$\begin{aligned}\Xi(z,k) \quad = \quad & \left(\frac{z_*}{z}\right)^2\left\{\left(\cosh|\alpha| - \frac{1}{kz_*}\sinh|\alpha|\right)\left[\frac{1}{k}\left(z_*^{-1} - z^{-1}\right) + \left(\frac{1}{k^2 z_* z} - 1\right)\sinh|\alpha|\right]\Delta^2_{\delta_M,*}(k)\right.\\ & + \left[\cosh|\alpha| + \frac{1}{kz}\sinh|\alpha|\right]\sinh|\alpha|\Delta^2_{\mu_M,*}(k)\\ & \left.\left[\cosh^2|\alpha| + \frac{1}{k}\left(z^{-1} - z_*^{-1}\right)\sinh|2\alpha| + \left(1 - \frac{2}{k^2 z z_*}\right)\sinh^2|\alpha|\right])\Xi_*(k)\right\}\end{aligned} \tag{C.1.36c}$$

where again $\Delta^2_{\delta_M,*}$, $\Delta^2_{\mu_M,*}$ and Ξ_* are the initial values, at a_* of this power spectrum coefficients. To "update" the coefficients we again use that $\hat{\gamma}_M(z,k) = -kz\hat{\mu}_M(z,k)$, so

$$(\Delta^2_{\delta_M,*})'(k) = \Delta^2_{\delta_M,*}(k), \qquad (\Delta^2_{\gamma_M,*})'(k) = (kz_*)^2\Delta^2_{\mu_M,*}(k), \qquad (\Xi_*)'(k) = -kz_*\Xi_*(k) \tag{C.1.37}$$

This way,

$$\hat{\delta}_M(a,k) = \frac{a}{a_*}\left[\left(\cosh|\alpha| - \frac{Ha_*}{k}\sinh|\alpha|\right)\hat{\delta}_{M,*}(k) + \frac{Ha_*}{k}\sinh|\alpha|\hat{\gamma}_{M,*}(k)\right] \tag{C.1.38a}$$

$$\hat{\gamma}_M(a,k) = -\frac{k}{Ha}\frac{a}{a_*}\left\{\left[\frac{H}{k}(a_* - a)\cosh|\alpha| + \left(\frac{H^2 a_* a}{k^2} - 1\right)\sinh|\alpha|\right]\hat{\delta}_{M,*}(k) - \frac{Ha_*}{k}\left[\cosh|\alpha| + \frac{Ha}{k}\sinh|\alpha|\right]\hat{\gamma}_{M,*}(k)\right\}, \tag{C.1.38b}$$

with the associated power coefficients being:

$$\begin{aligned}\Delta^2_{\delta_M}(a,k) \quad = \quad & \left(\frac{a}{a_*}\right)^2\left[\left(\cosh|\alpha| - \frac{Ha_*}{k}\sinh|\alpha|\right)^2\Delta^2_{\delta_M,*}(k) + \left(\frac{Ha_*}{k}\right)^2\sinh^2|\alpha|\Delta^2_{\gamma_M,*}(k)\right.\\ & \left.-2\frac{Ha_*}{k}\left(\frac{Ha_*}{k}\sinh^2|\alpha| - \frac{1}{2}\sinh|2\alpha|\right)\Xi_*(k)\right]\end{aligned} \tag{C.1.39a}$$

$$\begin{aligned}\Delta^2_{\gamma_M}(z,k) \quad = \quad & \left(\frac{k}{Ha}\right)^2\left(\frac{a}{a_*}\right)^2\left\{\left[\frac{H}{k}(a_* - a)\cosh|\alpha| + \left(\frac{H^2 a_* a}{k^2} - 1\right)\sinh|\alpha|\right]^2\Delta^2_{\delta_M,*}(k)\right.\\ & + \left(\frac{Ha_*}{k}\right)^2\left[\cosh|\alpha| + \frac{Ha}{k}\sinh|\alpha|\right]^2\Delta^2_{\gamma_M,*}(k)\\ & \left.-2\frac{Ha_*}{k}\left[\frac{H}{k}(a_* - a)\cosh|\alpha| + \left(\frac{H^2 a_* a}{k^2} - 1\right)\sinh|\alpha|\right]\left[\cosh|\alpha| + \frac{Ha}{k}\sinh|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{C.1.39b}$$

$$\begin{aligned}\Xi(a,k) \quad = \quad & -\frac{k}{Ha}\left(\frac{a}{a_*}\right)^2\left\{\left(\cosh|\alpha| - \frac{Ha_*}{k}\sinh|\alpha|\right)\left[\frac{H}{k}(a_* - a)\cosh|\alpha| + \left(\frac{H^2 a_* a}{k^2} - 1\right)\sinh|\alpha|\right]\Delta^2_{\delta_M,*}(k)\right.\\ & + \left(\frac{Ha_*}{k}\right)^2\left[\frac{1}{2}\sinh|2\alpha| + \frac{Ha}{k}\sinh^2|\alpha|\right]\Delta^2_{\gamma_M,*}(k)\\ & \left.-\frac{Ha_*}{k}\left[\cosh^2|\alpha| + \frac{H}{k}(a - a_*)\sinh|2\alpha| + \left(1 - \frac{2Ha_*}{k^2}\right)\sinh^2|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{C.1.39c}$$

C.1.3. Effective Equation of State $\omega_{eff} = \frac{1}{3}$. In this case the dominant fluid susceptible to perturbations is radiation. The gauge-invariant fractional density perturbation is given by

$$\delta_M = C_{eff}^{-1}\varepsilon a, \qquad \text{with} \quad C_{eff} = \bar{\rho}_{eff,0}a_0^4 \tag{C.1.40}$$

Again performing the Fourier transform on δ_M, we can write

$$\hat{\delta}_M(z,k) = C_{eff}^{-1}\sqrt{\frac{3}{C}}\frac{y}{z^2} = C_{eff}^{-1}\sqrt{\frac{C}{3}}\frac{\hat{\varphi}(z,k)}{z^2} \tag{C.1.41}$$

As for the power term,

$$\hat{\mu}_M(z,k) = \frac{1}{k}\frac{\partial}{\partial z}\hat{\delta}_M(z,k) = C_{eff}^{-1}\sqrt{\frac{C}{3}}z^{-2}\left[\hat{\phi}(z,k) - \frac{2}{zk}\hat{\varphi}(z,k)\right] \tag{C.1.42}$$

Applying eq. C.1.14, we have that

$$\hat{\delta}_M(z,k) = C_{eff}^{-1}\sqrt{\frac{C}{3}}z^{-2}\left[\cosh|\alpha|\hat{\varphi}_*(k) - \sinh|\alpha|\hat{\phi}_*(k)\right] \tag{C.1.43a}$$

$$\hat{\mu}_M(z,k) = C_{eff}^{-1}\sqrt{\frac{C}{3}}z^{-2}\left[\left(\hat{\phi}_*(k) - \frac{2}{kz}\hat{\varphi}_*(k)\right)\cosh|\alpha| - \left(\frac{2}{kz}\hat{\phi}_*(k) + \hat{\varphi}_*(k)\right)\sinh|\alpha|\right] \tag{C.1.43b}$$

We now find the expression of $\hat{\varphi}_*(k)$ and $\hat{\phi}_*(k)$ in terms of $\hat{\delta}_{M,*}(k)$ and $\hat{\mu}_{M,*}(k)$

$$\left\{\begin{array}{lcl} \hat{\delta}_{M,*}(k) &=& C_{eff}^{-1}\sqrt{\frac{C}{3}}z_*^{-2}\hat{\varphi}_*(k) \\ \hat{\mu}_{M,*}(k) &=& C_{eff}^{-1}\sqrt{\frac{C}{3}}z_*^{-2}\left[\hat{\phi}_*(k) - \dfrac{2}{kz_*}\hat{\varphi}_*(k)\right] \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} \hat{\varphi}_*(k) &=& C_{eff}\sqrt{\frac{C}{3}}z_*^2\hat{\delta}_{M,*}(k) \\ \hat{\phi}_*(k) &=& C_{eff}\sqrt{\frac{C}{3}}z_*^2\left(\hat{\mu}_{M,*}(k) + \dfrac{2}{kz_*}\hat{\delta}_{M,*}(k)\right) \end{array}\right. \tag{C.1.44}$$

It is now easy to see that the perturbations evolution is given by

$$\hat{\delta}_M(z,k) = \left(\frac{z_*}{z}\right)^2\left[\left(\cosh|\alpha| - \frac{2}{kz_*}\sinh|\alpha|\right)\hat{\delta}_{M,*}(k) - \sinh|\alpha|\hat{\mu}_{M,*}(k)\right] \tag{C.1.45a}$$

$$\hat{\mu}_M(z,k) = \left(\tfrac{z_*}{z}\right)^2\left\{\left[\tfrac{2}{k}\left(z_*^{-1} - z^{-1}\right)\cosh|\alpha| - \left(1 - \tfrac{4}{k^2zz_*}\right)\sinh|\alpha|\right]\hat{\delta}_{M,*}(k) + \left[\cosh|\alpha| + \tfrac{2}{kz}\sinh|\alpha|\right]\hat{\mu}_{M,*}(k)\right\} \tag{C.1.45b}$$

The associated power coefficients are the following:

$$\Delta^2_{\delta_M}(z,k) = \left(\frac{z_*}{z}\right)^4\left[\left(\cosh|\alpha| - \tfrac{2}{kz_*}\sinh|\alpha|\right)^2\Delta^2_{\delta_M,*}(k) + \sinh^2|\alpha|\Delta^2_{\mu_M,*}(k) + \left(\sinh|2\alpha| - \frac{4}{kz_*}\sinh^2|\alpha|\right)\Xi_*(k)\right] \tag{C.1.46a}$$

$$\begin{aligned} \Delta^2_{\mu_M}(z,k) \;=\; & \left(\frac{z_*}{z}\right)^4\left\{\left[\frac{2}{k}\left(z_*^{-1} - z^{-1}\right)\cosh|\alpha| - \left(1 - \frac{4}{k^2zz_*}\right)\sinh|\alpha|\right]^2\Delta^2_{\delta_M,*}(k)\right. \\ & + \left[\cosh|\alpha| + \frac{2}{kz}\sinh|\alpha|\right]^2\Delta^2_{\mu_M,*}(k) \\ & \left. +2\left[\frac{2}{k}\left(z_*^{-1} - z^{-1}\right)\cosh|\alpha| - \left(1 - \frac{4}{k^2zz_*}\right)\sinh|\alpha|\right]\left[\cosh|\alpha| + \frac{2}{kz}\sinh|\alpha|\right]\Xi_*(k)\right\} \end{aligned} \tag{C.1.46b}$$

$$\begin{aligned} \Xi(z,k) \;=\; & \left(\frac{z_*}{z}\right)^4\left\{\left(\cosh|\alpha| - \frac{2}{kz_*}\sinh|\alpha|\right)\left[\tfrac{2}{k}\left(z_*^{-1} - z^{-1}\right)\cosh|\alpha| - \left(1 - \frac{4}{k^2zz_*}\right)\sinh|\alpha|\right]\Delta^2_{\delta_M,*}(k)\right. \\ & - \left[\frac{1}{2}\sinh|2\alpha| + \frac{2}{kz}\sinh^2|\alpha|\right]\Delta^2_{\mu_M,*}(k) \\ & \left. + \left[\cosh^2|\alpha| + \tfrac{2}{k}\left(z^{-1} - z_*^{-1}\right)\sinh|2\alpha| + \left(1 - \tfrac{8}{k^2zz_*}\right)\sinh^2|\alpha|\right]\Xi_*(k)\right\} \end{aligned} \tag{C.1.46c}$$

where once again $\Delta^2_{\delta_M,*}$, $\Delta^2_{\mu_M,*}$ and Ξ are the initial values, at a_* of this power spectrum coefficients. We have to "update" the coefficients taking into account the variable changes, so that $\hat{\gamma}_M(z,k) = -kz\hat{\mu}_M(z,k)$, and

$$(\Delta^2_{\delta_M,*})'(k) = \Delta^2_{\delta_M,*}(k), \qquad (\Delta^2_{\gamma_M,*})'(k) = (kz_*)^2\Delta^2_{\mu_M,*}(k), \qquad (\Xi_*)'(k) = -kz_*\Xi_*(k) \tag{C.1.47}$$

Now,

$$\hat{\delta}_M(a,k) = \left(\frac{a}{a_*}\right)^2\left[\left(\cosh|\alpha| - \frac{2Ha_*}{k}\sinh|\alpha|\right)\hat{\delta}_{M,*}(k) + \frac{Ha_*}{k}\sinh|\alpha|\hat{\gamma}_{M,*}(k)\right] \tag{C.1.48a}$$

$$\hat{\gamma}_M(a,k) = \frac{k}{Ha}\left(\frac{a}{a_*}\right)^2\left\{\left[\frac{2H}{k}(a_* - a)\cosh|\alpha| - \left(1 - \frac{4Ha_*a}{k^2}\right)\sinh|\alpha|\right]\hat{\delta}_{M,*}(k) - \frac{Ha_*}{k}\left[\cosh|\alpha| + \frac{2Ha}{k}\sinh|\alpha|\right]\hat{\gamma}_{M,*}(k)\right\} \tag{C.1.48b}$$

As for the power coefficients:

$$\begin{aligned}\Delta^2_{\delta_M}(a,k) &= \left(\frac{a}{a_*}\right)^4\left[\left(\cosh|\alpha| - \frac{2Ha_*}{k}\sinh|\alpha|\right)^2\Delta^2_{\delta_M,*}(k) + \left(\frac{Ha_*}{k}\right)^2\sinh^2|\alpha|\Delta^2_{\gamma_M,*}(k)\right.\\ &\left. -\frac{Ha_*}{k}\left(\sinh|2\alpha| - \frac{4Ha_*}{k}\sinh^2|\alpha|\right)\Xi_*(k)\right]\end{aligned} \tag{C.1.49a}$$

$$\begin{aligned}\Delta^2_{\gamma_M}(a,k) &= \left(\frac{k}{Ha}\right)^2\left(\frac{a}{a_*}\right)^4\left\{\left[\frac{2H}{k}(a_*-a)\cosh|\alpha| - \left(1 - \frac{4H^2a_*a}{k^2}\right)\sinh|\alpha|\right]^2\Delta^2_{\delta_M,*}(k)\right.\\ &+\left(\frac{Ha_*}{k}\right)^2\left[\cosh|\alpha| + \frac{2Ha}{k}\sinh|\alpha|\right]^2\Delta^2_{\gamma_M,*}(k)\\ &\left.-2\frac{Ha_*}{k}\left[\frac{2H}{k}(a_*-a)\cosh|\alpha| - \left(1-\frac{4H^2a_*a}{k^2}\right)\sinh|\alpha|\right]\left[\cosh|\alpha| + \frac{2Ha}{k}\sinh|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{C.1.49b}$$

$$\begin{aligned}\Xi(z,k) &= \frac{k}{Ha}\left(\frac{a}{a_*}\right)^4\left\{\left(\frac{2Ha_*}{k}\sinh|\alpha| - \cosh|\alpha|\right)\left[\frac{2H}{k}(a_*-a)\cosh|\alpha| - \left(1-\frac{4H^2a_*a}{k^2}\right)\sinh|\alpha|\right]\Delta^2_{\delta_M,*}(k)\right.\\ &+\left(\frac{Ha_*}{k}\right)^2\left[\frac{1}{2}\sinh|2\alpha| + \frac{2Ha}{k}\sinh^2|\alpha|\right]\Delta^2_{\gamma_M,*}(k)\\ &\left.+\frac{Ha_*}{k}\left[\cosh^2|\alpha| + \tfrac{2Ha}{k}(a_*-a)\sinh|2\alpha| + \left(1-\frac{8H^2a_*a}{k^2}\right)\sinh^2|\alpha|\right]\Xi_*(k)\right\}\end{aligned} \tag{C.1.49c}$$

C.2. Pressureless Matter (Dust) ($\omega = 0$)

For this case, we consider the state equation $\omega = 0$, describing a Universe primarily composed by a pressureless matter fluid made up by dust. For this value of ω, equation 4.1.9 takes the following form:

$$a^2\frac{\partial^2\varepsilon}{\partial a^2} + \frac{3}{2}\left(a\frac{\partial\varepsilon}{\partial a} - \varepsilon\right) = 0 \tag{C.2.1}$$

Unlike the previous cases, eq. C.2.1 does not feature a laplacian term $\nabla^2\varepsilon$, the density evolution being only a function of a. To obtain a solution, it is natural to look for a polynomial one:

$$\left.\begin{aligned}\varepsilon &= Ba^n\\ \frac{\partial\varepsilon}{\partial a} &= nBa^{n-1}\\ \frac{\partial^2\varepsilon}{\partial a^2} &= n(n-1)Ba^{n-1}\end{aligned}\right\} \Rightarrow n(n-1) + \frac{3}{2}(n-1) = 0 \Rightarrow \left\{\begin{aligned} n&=1\\ n&=-\tfrac{3}{2}\end{aligned}\right. \tag{C.2.2}$$

This way, taking into account that the constant B needs not to be spatially constant, we have the following density evolution, considering a starting point a_r:

$$\varepsilon(\vec{x},a) = \widetilde{f}(\vec{x})\left(\frac{a}{a_r}\right) + \widetilde{g}(\vec{x})\left(\frac{a}{a_r}\right)^{-\frac{3}{2}} \tag{C.2.3}$$

where $\widetilde{f}(\vec{x})$ and $\widetilde{g}(\vec{x})$ are smooth functions of $\vec{x}$. Recalling that, from eq. 4.1.2,

$$\delta_M = \frac{\varepsilon}{C}, \qquad \text{with } C = \bar{\rho}_r a_r^3 \tag{C.2.4}$$

the density and power perturbations evolution will follow

$$\delta_M(a,\vec{x}) = f(\vec{x})\left(\frac{a}{a_r}\right) + g(\vec{x})\left(\frac{a}{a_r}\right)^{-\frac{3}{2}} \tag{C.2.5a}$$

$$\gamma_M(a,\vec{x}) := a\frac{\partial\delta_M}{\partial a} = f(\vec{x})\left(\frac{a}{a_r}\right) - \frac{3}{2}g(\vec{x})\left(\frac{a}{a_r}\right)^{-\frac{3}{2}} \tag{C.2.5b}$$

As before, we will want to write the evolution density in terms of the density and power terms:

$$\delta_{M,r}(\vec{x}) = \delta_M(\vec{x},a_r) = f(\vec{x}) + g(\vec{x}) \tag{C.2.6a}$$

$$\gamma_{M,r}(\vec{x}) = a_r\frac{\partial\delta_M}{\partial a}(\vec{x},a_0) = f(\vec{x}) - \frac{3}{2}g(\vec{x}) \tag{C.2.6b}$$

Solving the associated system, we get to

$$\delta_M(a,\vec{x}) = \delta_{M,r}(\vec{x})\left[\frac{3}{5}\left(\frac{a}{a_r}\right) + \frac{2}{5}\left(\frac{a}{a_r}\right)^{-3/2}\right] + \gamma_{M,r}(\vec{x})\left[\frac{2}{5}\left(\frac{a}{a_r}\right) - \frac{2}{5}\left(\frac{a}{a_r}\right)^{-3/2}\right] \tag{C.2.7a}$$

$$\gamma_M(a,\vec{x}) = \delta_{M,r}(\vec{x})\left[\frac{3}{5}\left(\frac{a}{a_r}\right) - \frac{3}{5}\left(\frac{a}{a_r}\right)^{-3/2}\right] + \gamma_{M,r}(\vec{x})\left[\frac{2}{5}\left(\frac{a}{a_r}\right) + \frac{3}{5}\left(\frac{a}{a_r}\right)^{-3/2}\right] \tag{C.2.7b}$$

The Fourier transform of $\delta_M(\vec{x},a)$ and $\gamma_M(\vec{x},a)$ simply consists in replacing $\delta_{M,r}(\vec{x})$ and $\gamma_{M,r}(\vec{x})$ by their respective transforms. This way we finally obtain the power spectrum associated with $\hat{\delta}_M$ and $\hat{\gamma}_M$, as well as the correlation term:

$$\Delta^2_{\delta_M}(a,k) = \langle\hat{\delta}_M(a,k)^2\rangle = \frac{1}{25}\left\{\Delta^2_{\delta_M,r}(k)\left[9\left(\frac{a}{a_r}\right)^2 + 12\left(\frac{a}{a_r}\right)^{-1/2} + 4\left(\frac{a}{a_r}\right)^{-3}\right]\right.$$
$$\left.+\Delta^2_{\gamma_M,r}(k)\left[4\left(\frac{a}{a_r}\right)^2 - 8\left(\frac{a}{a_r}\right)^{-1/2} + 4\left(\frac{a}{a_r}\right)^{-3}\right] + \Xi_r(k)\left[12\left(\frac{a}{a_r}\right)^2 - 4\left(\frac{a}{a_r}\right)^{-1/2} - 8\left(\frac{a}{a_r}\right)^{-3}\right]\right\} \tag{C.2.8a}$$

$$\Delta^2_{\gamma_M}(a,k) = \langle\hat{\gamma}_M(a,k)^2\rangle = \frac{1}{25}\left\{\Delta^2_{\delta_M,r}(k)\left[9\left(\frac{a}{a_r}\right)^2 - 18\left(\frac{a}{a_r}\right)^{-1/2} + 9\left(\frac{a}{a_r}\right)^{-3}\right]\right.$$
$$\left.+\Delta^2_{\gamma_M,r}(k)\left[4\left(\frac{a}{a_r}\right)^2 + 12\left(\frac{a}{a_r}\right)^{-1/2} + 9\left(\frac{a}{a_r}\right)^{-3}\right] + \Xi_r(k)\left[12\left(\frac{a}{a_r}\right)^2 + 6\left(\frac{a}{a_r}\right)^{-1/2} - 18\left(\frac{a}{a_r}\right)^{-3}\right]\right\} \tag{C.2.8b}$$

$$\Xi(a,k) = \langle\hat{\delta}_M(a,k)\hat{\gamma}_M(a,k)\rangle = \frac{1}{25}\left\{\Delta^2_{\delta_M,r}(k)\left[9\left(\frac{a}{a_r}\right)^2 - 3\left(\frac{a}{a_r}\right)^{-1/2} - 6\left(\frac{a}{a_r}\right)^{-3}\right]\right.$$
$$\left.+\Delta^2_{\gamma_M,r}(k)\left[4\left(\frac{a}{a_r}\right)^2 + 2\left(\frac{a}{a_r}\right)^{-1/2} - 6\left(\frac{a}{a_r}\right)^{-3}\right] + \Xi_r(k)\left[12\left(\frac{a}{a_r}\right)^2 + \left(\frac{a}{a_r}\right)^{-1/2} + 12\left(\frac{a}{a_r}\right)^{-3}\right]\right\} \tag{C.2.8c}$$

C.3. Radiation ($\omega = \frac{1}{3}$)

We now turn to a Universe dominated by radiation, with a equation of state $\omega = \frac{1}{3}$. This way, eq. 4.1.9 can be written as

$$a^2\frac{\partial^2\varepsilon}{\partial a^2} + 2\left(a\frac{\partial\varepsilon}{\partial a} - \varepsilon\right) - \frac{1}{C}a^2\nabla^2\varepsilon = 0 \tag{C.3.1}$$

Dividing by a^2, eq. C.3.1 takes the following form:

$$\frac{\partial^2\varepsilon}{\partial a^2} + \frac{2}{a}\frac{\partial\varepsilon}{\partial a} - \frac{2}{a^2}\varepsilon - \frac{1}{C}\nabla^2\varepsilon = 0 \tag{C.3.2}$$

It is easy to take notices that, of the three cases studied so far, this is the one in which the evolution equation takes the more complicated form. Trying to simplify it, we define

$$u := \frac{a}{\sqrt{C}}, \tag{C.3.3}$$

so

$$\frac{\partial^2\varepsilon}{\partial u^2} + \frac{2}{u}\frac{\partial\varepsilon}{\partial z} - \frac{2}{u^2}\varepsilon - \nabla^2\varepsilon = 0, \tag{C.3.4}$$

as $\frac{\partial}{\partial a} = C^{-\frac{1}{2}}\frac{\partial}{\partial u}$ and $\frac{\partial^2}{\partial a^2} = C^{-2}\frac{\partial^2}{\partial u^2}$.
We can multiply eq. C.3.4 by u^2, resulting in:

$$u^2\frac{\partial^2\varepsilon}{\partial u^2} + 2u\frac{\partial\varepsilon}{\partial z} - (2 + u^2\nabla^2)\varepsilon = 0 \tag{C.3.5}$$

Translating this expression into the Fourier modes $\hat{\varepsilon}(u,k)$ we get to

$$u^2\frac{\partial^2\hat{\varepsilon}}{\partial u^2} + 2u\frac{\partial\hat{\varepsilon}}{\partial u} + (u^2k^2 - 2)\hat{\varepsilon} = 0 \tag{C.3.6}$$

We have a Sturm-Liouville equation, with the following solution:

$$\hat{\varepsilon}(u,k) = \widetilde{A}(k)j_1(ku) + \widetilde{B}(k)y_1(ku), \tag{C.3.7}$$

where j_1 and y_1 are the spherical Bessel functions of the first and second kind, respectively. Luckily for us, their expression in terms of elemental functions is not very complicated:

$$j_1(x) := -\frac{\mathrm{d}j_0(x)}{\mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{C.3.8a}$$

$$y_1(x) := -\frac{\mathrm{d}y_0(x)}{\mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{\cos x}{x}\right) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \tag{C.3.8b}$$

Defining $z = uk$, we can rearrange the solution to the following expression:

$$\begin{aligned}\hat{\varepsilon}(z,k) &= \frac{1}{z^2}\left[\bar{A}(k)\sin(z-z_i)+\bar{B}(k)\cos(z-z_i)\right]-\frac{1}{z}\left[\bar{A}(k)\cos(z-z_i)-\bar{B}(k)\sin(z-z_i)\right]\\ &= \frac{1}{z^2}\left[\bar{A}(k)+z\bar{B}(k)\right]\sin(z-z_i)+\frac{1}{z^2}\left[\bar{B}(k)-z\bar{A}(k)\right]\cos(z-z_i)\end{aligned} \tag{C.3.9}$$

Using eq. 4.1.2, $\delta_M = C^{-1}a\varepsilon = C^{\frac{1}{2}}u\varepsilon = C^{\frac{1}{2}}k^{-1}z\varepsilon$ the density perturbations evolution, as well as their velocity, can be described by:

$$\hat{\delta}_M(z,k) = \left[\frac{A(k)}{z}+B(k)\right]\sin(z-z_i)+\left[\frac{B(k)}{z}-A(k)\right]\cos(z-z_i) \tag{C.3.10a}$$

$$\hat{\mu}_M(z,k) := \frac{\partial}{\partial z}\hat{\delta}_M(z,k) = \left[A(k)-\frac{A(k)}{z^2}-\frac{B(k)}{z}\right]\sin(z-z_i)+\left[\frac{A(k)}{z}+B(k)-\frac{B(k)}{z^2}\right]\cos(z-z_i) \tag{C.3.10b}$$

As in other cases, we find the density and power initial terms:

$$\hat{\delta}_{M,i}(k) := \hat{\delta}_M(z_i,k) = \frac{1}{z_0}B(k)-A(k) \tag{C.3.11a}$$

$$\hat{\mu}_{M,i}(k) := \hat{\mu}_M(z_i,k) = \frac{\bar{A}(k)}{z}+\bar{B}(k)-\frac{B(k)}{z^2} \tag{C.3.11b}$$

In which we can express the functions $A(k)$ and $B(k)$:

$$A(k) = \left(\frac{1}{z_i^2}-1\right)\hat{\delta}_{M,i}(k)+\frac{1}{z_i}\hat{\mu}_{M,i}(k), \qquad B(k) = \frac{1}{z_i}\hat{\delta}_{M,i}(k)+\hat{\mu}_{M,i}(k) \tag{C.3.12}$$

After some tedious calculations, we get to the following final result:

$$\hat{\delta}_M(z,k) = \left[\alpha(z)\hat{\delta}_{M,i}(k)+\beta(z)\hat{\mu}_{M,i}(k)\right]\sin(z-z_i)+\left[\lambda(z)\hat{\delta}_{M,i}(k)+\gamma(z)\hat{\mu}_{M,i}(k)\right]\cos(z-z_i) \tag{C.3.13a}$$

$$\hat{\mu}_M(z,k) = \left[a(z)\hat{\delta}_{M,i}(k)+b(z)\hat{\mu}_{M,i}(k)\right]\cos(z-z_i)+\left[c(z)\hat{\delta}_{M,i}(k)+d(z)\hat{\mu}_{M,i}(k)\right]\sin(z-z_i) \tag{C.3.13b}$$

where the following auxiliary functions have been defined:

$$\alpha(z) := \frac{1}{z_i}-\frac{1}{z}+\frac{1}{zz_i^2}, \qquad \beta(z) := 1+\frac{1}{zz_i}, \qquad \lambda(z) := 1+\frac{1}{zz_i}-\frac{1}{z_i^2}, \qquad \gamma(z) := \frac{1}{z}-\frac{1}{z_i} \tag{C.3.14a}$$

$$\begin{aligned}a(z) &:= \frac{1}{z_i}-\frac{1}{z}+\frac{1}{zz_i^2}-\frac{1}{z^2z_i}, \quad b(z) := 1+\frac{1}{zz_i}-\frac{1}{z^2},\\ c(z) &:= -1-\frac{1}{z^2}-\frac{1}{z_i^2}-\frac{1}{zz_i}-\frac{1}{z^2z_i^2}, \quad d(z) := \frac{1}{z_i}-\frac{1}{z}-\frac{1}{z^2z_i}\end{aligned} \tag{C.3.14b}$$

As a curiosity, the evolution of the perturbations could also be expressed using a matrix notation:

$$\begin{pmatrix}\hat{\delta}_M(z,k)\\ \hat{\mu}_M(z,k)\end{pmatrix} = \underbrace{\left\{\begin{bmatrix}\alpha(z) & \beta(z)\\ c(z) & d(z)\end{bmatrix}\sin(z-z_i)+\begin{bmatrix}\lambda(z) & \gamma(z)\\ a(z) & b(z)\end{bmatrix}\cos(z-z_i)\right\}}_{T(z,z_i)}\begin{pmatrix}\hat{\delta}_{M,i}(k)\\ \hat{\mu}_{M,i}(k)\end{pmatrix}, \tag{C.3.15}$$

We are now ready to compute the power spectrum coefficients for $\hat{\delta}_M$ and $\hat{\mu}_M$, defined in the same way as in the other cases:

$$\begin{aligned}\Delta^2_{\delta_M}(z,k) &= \left[\alpha(z)\sin(z-z_i)+\lambda(z)\cos(z-z_i)\right]^2\Delta^2_{\delta_M,i}(k)\\ &+ \left[\beta(z)\sin(z-z_i)+\gamma(z)\cos(z-z_i)\right]^2\Delta^2_{\mu_M,i}(k)\\ &+ 2\left[\alpha(z)\sin(z-z_i)+\lambda(z)\cos(z-z_i)\right]\left[\beta(z)\sin(z-z_i)+\gamma(z)\cos(z-z_i)\right]\Xi_i(k)\end{aligned} \tag{C.3.16a}$$

$$\begin{aligned}\Delta^2_{\mu_M}(z,k) &= \left[a(z)\cos(z-z_i)+c(z)\sin(z-z_i)\right]^2\Delta^2_{\delta_M,i}(k)\\ &+ \left[b(z)\cos(z-z_i)+d(z)\sin(z-z_i)\right]^2\Delta^2_{\mu_M,i}(k)\\ &+ 2\left[a(z)\cos(z-z_i)+c(z)\sin(z-z_i)\right]\left[b(z)\cos(z-z_i)+d(z)\sin(z-z_i)\right]\Xi_i(k)\end{aligned} \tag{C.3.16b}$$

$$\begin{aligned}\Xi(z,k) &= \left[\alpha(z)\sin(z-z_i)+\lambda(z)\cos(z-z_i)\right]\left[a(z)\cos(z-z_i)+c(z)\sin(z-z_i)\right]\Delta^2_{\delta_M,i}(k)\\ &+ \left[\beta(z)\sin(z-z_i)+\gamma(z)\cos(z-z_i)\right]\left[b(z)\cos(z-z_i)+d(z)\sin(z-z_i)\right]\Delta^2_{\mu_M,i}(k)\\ &+ \{\left[\alpha(z)\sin(z-z_i)+\lambda(z)\cos(z-z_i)\right]\left[b(z)\cos(z-z_i)+d(z)\sin(z-z_i)\right]\\ &\quad+\left[\beta(z)\sin(z-z_i)+\gamma(z)\cos(z-z_i)\right]\left[a(z)\cos(z-z_i)+c(z)\sin(z-z_i)\right]\}\,\Xi_i(k)\end{aligned} \tag{C.3.16c}$$

where $\Delta^2_{\delta_M,i}$, $\Delta^2_{\mu_M,i}$ and Ξ_i are the initial values, at a_i of this power spectrum coefficients. Again, the velocity parameter $\hat{\mu}_M$ is not defined with respect to $\ln a$, but rather to z, so:

$$\hat{\mu}_M = \frac{\partial \hat{\delta}_M}{\partial z} = \left(\frac{\partial z}{\partial a}\right)^{-1} \frac{\partial \hat{\delta}_M}{\partial a} = \frac{a}{z}\frac{\partial \hat{\delta}_M}{\partial a} = \frac{1}{z}\frac{\partial \hat{\delta}_M}{\partial \ln a} = \frac{\hat{\gamma}_M}{z} \tag{C.3.17}$$

We again have to redefine some of the power spectrum coefficients to their standard expression:

$$P'_{\delta_M}(k) = \Delta^2_{\delta_M}(k), \qquad P'_{\gamma_M}(k) = z^2\Delta^2_{\mu_M}(k), \qquad (\Xi)'(k) = -z\Xi(k) \tag{C.3.18}$$

This way,

$$\hat{\delta}_M(z,k) = \Big[\alpha(z)\hat{\delta}_{M,i}(k) + z_i^{-1}\beta(z)\hat{\gamma}_{M,i}(k)\Big]\sin(z-z_i) + \Big[\lambda(z)\hat{\delta}_{M,i*}(k) + z_i^{-1}\gamma(z)\hat{\gamma}_{M,i}(k)\Big]\cos(z-z_i) \tag{C.3.19a}$$

$$\hat{\gamma}_M(z,k) = z\left\{\Big[a(z)\hat{\delta}_{M,i}(k) + z_i^{-1}b(z)\hat{\gamma}_{M,i}(k)\Big]\cos(z-z_i) + \Big[c(z)\hat{\delta}_{M,i}(k) + z_i^{-1}d(z)\hat{\gamma}_{M,i}(k)\Big]\sin(z-z_i)\right\} \tag{C.3.19b}$$

with the power coefficients being:

$$\begin{aligned}\Delta^2_{\delta_M}(z,k) &= [\alpha(z)\sin(z-z_i) + \lambda(z)\cos(z-z_i)]^2\,\Delta^2_{\delta_M,i}(k)\\ &+ z_i^{-2}[\beta(z)\sin(z-z_i) + \gamma(z)\cos(z-z_i)]^2\,\Delta^2_{\gamma_M,i}(k)\\ &- 2z_i^{-1}[\alpha(z)\sin(z-z_i) + \lambda(z)\cos(z-z_i)]\,[\beta(z)\sin(z-z_i) + \gamma(z)\cos(z-z_i)]\,\Xi_i(k)\end{aligned} \tag{C.3.20a}$$

$$\begin{aligned}\Delta^2_{\gamma_M}(z,k) &= z^2\Big\{[a(z)\cos(z-z_i) + c(z)\sin(z-z_i)]^2\,\Delta^2_{\delta_M,i}(k)\\ &+ z_i^{-2}[b(z)\cos(z-z_i) + d(z)\sin(z-z_i)]^2\,\Delta^2_{\gamma_M,i}(k)\\ &- 2z_i^{-1}[a(z)\cos(z-z_i) + c(z)\sin(z-z_i)]\,[b(z)\cos(z-z_i) + d(z)\sin(z-z_i)]\,\Xi_i(k)\Big\}\end{aligned} \tag{C.3.20b}$$

$$\begin{aligned}\Xi(z,k) &= -z\Big\{[\alpha(z)\sin(z-z_i) + \lambda(z)\cos(z-z_i)]\,[a(z)\cos(z-z_i) + c(z)\sin(z-z_i)]\,\Delta^2_{\delta_M,i}(k)\\ &+ z_i^{-2}[\beta(z)\sin(z-z_i) + \gamma(z)\cos(z-z_i)]\,[b(z)\cos(z-z_i) + d(z)\sin(z-z_i)]\,\Delta^2_{\gamma_M,i}(k)\\ &- z_i^{-1}\{[\alpha(z)\sin(z-z_i) + \lambda(z)\cos(z-z_i)]\,[b(z)\cos(z-z_i) + d(z)\sin(z-z_i)]\\ &+ [\beta(z)\sin(z-z_i) + \gamma(z)\cos(z-z_i)]\,[a(z)\cos(z-z_i) + c(z)\sin(z-z_i)]\}\,\Xi_i(k)\}\end{aligned} \tag{C.3.20c}$$

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